The Fundamental Properties of Autoregressive Models with Deterministically Varying Coefficients

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Abstract

The paper examines the problem of representing the dynamics of low order autoregressive models with variable coefficients. The existing literature computes the forecasts of the series from a recursion relation. Instead, we provide their linearly independent solutions. Our solution formulas enable us to derive the fundamental properties of these processes, and obtain explicit expressions for the optimal predictors. We illustrate our methodology and results with a few classic examples amenable to time varying treatment, e.g., periodic, cyclical, and models subject to multiple structural breaks.

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1. Introduction

The constancy of the parameters assumption made in the specification of time series econometric models has been the subject of criticism for a long time. It is argued that the assumption is inappropriate in the face of changing institutions and a dynamically responding economic policy. These evolving factors cause the coefficients values characterizing economic relationships to change over time. Partly to respond to the criticism and partly motivated by the desire to construct dynamic models, econometricians have developed an arsenal of powerful methods that attempt to capture the evolving nature of our economy. Such frameworks include processes which contain multiple abrupt breaks, and periodic and cyclical autoregressive (AR) models.

A methodology is presented in this paper for analyzing time varying systems[1] which is also applicable to the three aforementioned processes. A technique is set forth for examining the periodic AR model, which overcomes the usual requirement of expressing it in a vector AR (VAR) form.

The first attempts to develop theories for time varying models, made in the 1960's, were based on a recursive approach (Whittle, 1965) and on evolutionary spectral representations (Abdrabbo and Priestley, 1967). Rao (1970) used the method of weighted least squares to estimate an AR model with variable coefficients. Despite nearly half a century of research work, the great advances, and the widely recognized importance of time varying structures, the bulk of econometric models have constant coefficients. There is a lack of a general theory that can be employed to systematically explore their time series properties. Granger in some of his last contributions highlighted the importance of the topic (see, Granger 2007, and 2008).

There is a general agreement that the main obstacle to progress is the lack of a universally applicable method yielding a closed form solution to stochastic time varying difference equations. The present paper is part of a research program aiming to produce and utilize closed form solutions to AR processes with deterministically time varying coefficients (DTV-AR). Our methodology attempts to trace the path of these changing coefficients. To be specific, in the time series literature, there is no method for finding the p linearly independent solutions that we need in order to obtain the explicit representation (or general solution) of the TV-AR model of order p. To keep the exposition tractable and reveal its practical significance we work with low order specifications.

The main part of the paper begins with Subsection 2.2, where we state the second order difference equation with variable coefficients, which is our main object of inquiry. We start by writing this equation in a more efficient way as an infinite linear system. The next step is to define the matrix of coefficients, called the fundamental solution matrix, associated with the system representation. This matrix is the workhorse of our research and it is derived step by step from the time varying coefficients of the difference equation.

The reader will have noticed that we have moved the goalposts, paradoxically against us, from obtaining a solution for a time varying (low order) difference equation, to solving an infinite linear system. The reason is that the solution of such infinite

[1] We will refer to the AR models with time varying coefficients (TV-AR) as time varying models.

systems has been made possible recently, due to an extension of the standard Gaussian elimination, called the infinite Gaussian elimination (Paraskevopoulos, 2012; see also Paraskevopoulos, 2014). Applying this infinite extension algorithm, we obtain the fundamental solutions, which take explicit forms in terms of the determinants of the fundamental solution matrix.

Subsection 2.3 contains the main theoretical result of the paper. Pursuing the conventional route followed by the differential and difference equations literature, we construct the general solution by finding its two parts, the homogeneous one and a particular part. It is expressed as Theorem 1 and its proof is given in Appendix A. The coefficients in this solution are expressed as determinants of tridiagonal matrices. The second order properties of the TV-AR process can easily be deduced from the general solution. An additional benefit of these solutions is the facility with which linear prediction can be produced. This allows us to provide a thorough description of time varying models by deriving: first, multistep ahead forecasts, the associated forecast error and the mean square error; second, the first two unconditional moments of the process and its covariance structure. In related works we provide results for the p-th order and the more general ascending order (see, for example, Paraskevopoulos and Karanasos, 2021, and Karanasos et al., 2022). Our method is a natural extension of the first order solution formula. It also includes the linear difference equation with constant coefficients (see, for example, Karanasos, 2001) as a special case.

The next two Sections of the paper, 3 and 4, apply our theoretical framework to a few classic time series models, which are obvious candidates for a time varying treatment. Linear systems with time dependent coefficients are not only of interest in their own right, but, because of their connection with periodic models and time series data which are subject to structural breaks. They also provide insight into these processes as well. Viewing a periodic AR (PAR) formulation as a TV model clearly obviates the need for VAR analysis. For surveys and a review of some important aspects of PAR processes see Franses (1996b), Ghysels and Osborn (2001), Franses and Paap (2004), and Hurd and Miamee (2007). The authoritative studies by Osborn (1988), Birchenhall et al. (1989), and Osborn and Smith (1989) applied these models to consumption. Del Barrio Castro and Osborn (2008) pointed out that "*despite the attraction of PAR models from the perspective of economic decision making in a seasonal context, the more prominent approach of empirical workers is to assume that the AR coefficients, except for the intercept, are constant over the seasons of the year*".[2]

Despite the recognized importance of periodic processes for economics there have been few attempts to investigate their time series properties (see, among others, Franses, 1994, Franses, 1996a, Lund and Basawa, 2000, Franses and Paap, 2005). Tiao and Grupe (1980) and Osborn (1991) analyzed these models by converting them into a VAR process with constant parameters. In this paper we develop a general theory that can be employed to systematically explore the fundamental properties of the

[2] Del Barrio Castro and Osborn (2008, 2012) (see the references therein for this stream of important research; see also Taylor, 2002, 2003 and 2005) test for seasonal unit roots in integrated PAR models.

periodic formulation. We remain within the univariate framework and we look upon the PAR model as a stochastic difference equation with time varying (albeit periodically varying) parameters.

Although some theoretical analysis of periodic specifications was carried out by the aforementioned studies the investigation of their fundamental properties appears to have been limited to date. Cipra and Tlustý (1987), Anderson and Vecchia (1993), Adams and Goodwin (1995), Shao (2008), and Tesfaye et al. (2011) discuss parameter estimation and asymptotic properties of periodic autoregressive moving average (PARMA) specifications. Bentarzi and Hallin (1994) and McLeod (1994) derive invertibility conditions and diagnostic checks for such processes. Lund and Basawa (2000) develop a recursive scheme for computing one-step ahead predictors for PARMA specifications, and compute multi-step-ahead predictors recursively from the one-step-ahead predictions. Anderson et al. (2013) develop a recursive forecasting algorithm for periodic models. We derive explicit formulas that allow the analytic calculation of the multi-step-ahead predictors.

We begin Subsection 3.1 with a PAR(2) model. We limit our analysis to a low order to save space and also since Franses (1996a) has documented that low order PAR specifications often emerge in practice. First, we formulate it as a TV model; then, we express its fundamental solution matrix as a block Toeplitz matrix. This representation enables us to establish an explicit formula for the general solution in terms of the determinant of such a block matrix. The result is presented in Proposition 3, which is the equivalent to Theorem 1 with the incorporation of the seasonal effects. That is, by taking account of seasons and periodicities, we obtain the general solution, by constructing its homogeneous and particular parts and then adding them up. In Subsection 3.2, we turn our attention to a different type of seasonality, namely the cyclical AR (CAR) model and we provide its solution.

Section 4 is an application of the time varying framework to time series subject to multiple structural breaks. We employ a technique analogous to the one used in Section 3 on the PAR formulation. In particular, we express the fundamental solutions of the AR(2) model with r abrupt breaks, as determinants of block tridiagonal matrices. Again, we are able to obtain the general solution by finding and adding the homogeneous and particular solutions.

One of the advantages of our time varying framework is that we can trace the entire path of the series under consideration. In Section 5, we employ this information feature to derive the fundamental properties of the various TV-AR processes. For example, simplified closed-form expressions of the multi-step forecast error variances are derived for time series when low order PAR models adequately describe the data. These formulae allow a fast computation of the multi-step-ahead predictors. Finally, Section 6 concludes.

2. Time Varying AR Models

2.1 Preliminaries and Purpose of Analysis

Notation

Throughout the paper we adhere to the following conventions: $(Z_{>0})$ Z and $(R_{>0})$ R stand for the sets of (positive) integers, and (positive) real numbers, respectively. Matrices and vectors are denoted by upper and lower case boldface symbols, respectively. For square matrices $X = [x_{ij}]_{i,j=1,\dots,k} \in R^{k\times k}$ using standard notation, $\det(X)$ or $|X|$ denotes the determinant of matrix X and $\mathrm{adj}(\mathrm{X})$ its adjoint matrix.

The latest time-point of the observed random variables is denoted by $t\in Z$, and $k\in Z_{>0}$ such that at time $\tau \stackrel{\text{def}}{=} \tau_k = t - k$ information is given.
Let the triple (Ω, *F*, *P*) denote a probability space. Let also L_2(Ω, *F*, *P*) (in short L_2) stand for the Hilbert space of real random variables with finite first and second moments defined on (Ω, *F*, *P*).

The Problem

The solution of the second order linear difference equation with non variable coefficients is the building block for the extension of the well known closed form solution of the first order to the p-th order time varying equation. As noted by Sydsaeter et al. (2008), in their classic text (Further Mathematics for Economic Analysis, p. 403), in the case of second order homogeneous linear difference equations with variable coefficients:

"*There is no universally applicable method of discovering the two linearly independent solutions that we need in order to find the general solution of the equation.*"

We can identify two lines of inquiry that can be pursued to solve linear difference equations with time varying coefficients. Searching for a solution, one can follow either of the following two paths. The first is to develop an analogous method to the standard one that exists for the linear p-th order difference equation with constant coefficients: find the eigenvalues, solve the characteristic equation, and obtain the closed form. The second line of research searches for the generalization of the closed form formula that exists for first order time varying difference equations. Here, the way to proceed is to make up a conjecture and try to prove it by induction. The two strands of the literature have taken important steps, but have not provided us with a general solution method that we can apply; the existing results lack generality and applicability. To be more specific, the research problem we face is that there is a lack of a universally applicable method yielding a closed form solution to stochastic higher order difference equations with time dependent coefficients.

A general method for solving infinite linear systems with row-finite coefficient matrices[3] has recently been established by Paraskevopoulos (2012). It is a modified

[3]A row-finite matrix is an infinite matrix, each row of which comprises a finite number of non-zero entries.

version of the standard Gauss-Jordan elimination method implemented under a right pivot strategy, called infinite Gauss-Jordan elimination. Expressing the linear difference equation of second order with time dependent coefficients as an infinite linear system, the Gaussian elimination part of the method is directly applicable. It generates two linearly independent homogeneous solution sequences. The general term of each solution sequence turns out to be a continuant determinant. The general solutions of the homogeneous and nonhomogeneous difference equation are expressible as a single Hessenbergian, that is, a determinant of a lower Hessenberg matrix (see Karanasos *et al*., 2022). The results in Paraskevopoulos and Karanasos (2021) afford an easy means of finding, for a given lower Hessenberg matrix, its ordinary expansion in non-determinant form. These results are extendible to the solution of the *p*-th and ascending order time varying linear difference equations in terms of a single Hessenbergian (see Paraskevopoulos and Karanasos, 2021). This makes it possible to introduce, in the above cited reference, a unified theory for time varying models.

2.2 Fundamental Solution Matrices

The main theoretical contribution of this Section is the development of a method that provides the closed form of the general solution to a TV-AR(2) model.
Next we give the main definition that we will use in the rest of the paper. Consider a second order stochastic difference equation with time dependent coefficients, which is equivalent to the time varying AR(2) process, given by

$$y_t = \varphi_0(t) + \varphi_1(t) y_{t-1} + \varphi_2(t) y_{t-2} + \varepsilon_t, \tag{1}$$

where $\{\varepsilon_t, t \in Z\}$ is a sequence of zero mean serially uncorrelated random variables defined on $L_2(\Omega, F, P)$ while $\{\varepsilon_t\}_t$ is a zero mean random process (that is $E(\varepsilon_t) = 0$) such that $E(\varepsilon_t \varepsilon_\tau) = 0$ for $t \neq \tau$ (uncorrelatedness condition), $E(\varepsilon_t | y_\tau,\ \tau < t) = 0$ for all t (that is, $\{\varepsilon_t\}$ is a martingale difference sequence relative to $\{y_t\}$), and the time varying variance σ^2 (t) is non-zero and bounded, that is $0 < \sigma^2(t) < M < \infty$, for all t and some $M \in R_{>0}$. The above conditions guarantee that $\varepsilon_t \in L_2$ and $\varepsilon_t \perp \varepsilon_\tau$ (ε_t, ε_τ are orthogonal) whenever $t \neq \tau$.

Remark 1 *We have relaxed the assumption of homoscedasticity (see also, among others, Karanasos et. al., 2014, Canepa et al., 2022 and Karanasos et al., 2022), which is likely to be violated in practice and allow* ε_t *to follow, for example, a periodical GARCH type of process (see, Bollerslev and Ghysels, 1996).*

The relation between the process under consideration and its innovations is essentially described by the Wold-Cramér decomposition (see Section 5.2), which is the main analytical tool for studying the asymptotic efficiency of the model. In this case, the latest time-point of the observed random variables, denoted here by τ, moves to the remote past $\tau = t - k \to -\infty \; or \; k \to \infty$, while the forecast time-point, denoted here by *t*, is kept fixed.

The fundamental solution sequence, and in general all the solution sequences, must necessarily be functions of the independent variable *t*, so as to satisfy eq. (1).

Our intermediate objective is to obtain the fundamental solution matrix, denoted below by $\mathbf{\Phi}_{t,\tau}$, which is associated with our stochastic difference equation (1); the $\mathbf{\Phi}_{t,\tau}$ matrix will be derived from the time varying coefficients of eq. (1). The best way to appreciate the representation of the fundamental solution matrix is to view the stochastic difference equation as a linear system. We carry out this construction below. Once we have this stepping stone in place, then we can pursue our ultimate objective, by computing the determinants of the $\mathbf{\Phi}_{t,\tau}$, which will give us the linearly independent solutions sequences to the difference equation.
Eq. (1) written as

$$\varphi_2(t)y_{t-2} + \varphi_1(t)y_{t-1} - y_t = -[\varphi_0(t) + \varepsilon_t], \tag{2}$$

takes the infinite row (and column)-finite system form

$$\mathbf{\Phi} \cdot \mathbf{y} = -\boldsymbol{\varphi} - \boldsymbol{\varepsilon}, \tag{3}$$

where

$$\mathbf{\Phi}=\begin{pmatrix} \varphi_2(\tau+1) & \varphi_1(\tau+1) & -1 & 0 & 0 & 0 & \cdots \\ 0 & \varphi_2(\tau+2) & \varphi_1(\tau+2) & -1 & 0 & 0 & \cdots \\ 0 & 0 & \varphi_2(\tau+3) & \varphi_1(\tau+3) & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

(row-finite is an infinite matrix whose rows have finite non zero elements) and

$$\mathbf{y} = \begin{pmatrix} y_{\tau-1} \\ y_\tau \\ y_{\tau+1} \\ y_{\tau+2} \\ y_{\tau+3} \\ y_{\tau+4} \\ \vdots \end{pmatrix}, \boldsymbol{\varphi} = \begin{pmatrix} \varphi_0(\tau+1) \\ \varphi_0(\tau+2) \\ \varphi_0(\tau+3) \\ \vdots \end{pmatrix}, \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{\tau+1} \\ \varepsilon_{\tau+2} \\ \varepsilon_{\tau+3} \\ \vdots \end{pmatrix}$$

(recall that τ=t-k). The system representation results from the values that the coefficients take in successive time periods. The equivalence of eqs. (2) and (3) follows from the fact that the i-th equation in (3), as a result of the multiplication of the i -th row of $\mathbf{\Phi}$ by the column of ys equated to $-[\varphi_0(\tau+i) + \varepsilon_{\tau+i}]$, is equivalent to eq. (2), as of time τ+i. The $\mathbf{\Phi}$ matrix in eq. (3) can be partitioned as

$$\mathbf{\Phi} = (\mathbf{P}|\mathbf{C})$$

where

$$\mathbf{P} = \begin{pmatrix} \varphi_2(\tau+1) & \varphi_1(\tau+1) \\ 0 & \varphi_2(\tau+2) \\ 0 & 0 \\ \vdots & \vdots \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots \\ \varphi_1(\tau+2) & -1 & 0 & 0 & \cdots \\ \varphi_2(\tau+3) & \varphi_1(\tau+3) & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

That is, **P** consists of the first 2 columns of **Φ** and the *j*-th column of **C**, $j = 1,2, \ldots$, is the *(2+j)*-th column of **Φ**. We will denote the 2nd column of the $k \times 2$ top submatrix of the matrix **P** by $\boldsymbol{\varphi}_{t,\tau}$:

$$(\boldsymbol{\varphi}_{t,\tau})' = (\varphi_1(\tau+1), \quad \varphi_2(\tau+2), \quad 0, \quad \ldots \quad ,0).$$

The $k \times (k-1)$ top submatrix of matrix **C** is called the core solution matrix and is denoted as

$$\mathbf{C}_{t,\tau} = \begin{pmatrix} -1 & & & & \\ \varphi_1(\tau+2) & -1 & & & \\ \varphi_2(\tau+3) & \varphi_1(\tau+3) & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_2(t-1) & \varphi_1(t-1) & -1 \\ & & & \varphi_2(t) & \varphi_1(t) \end{pmatrix} \tag{4}$$

(here and in what follows empty spaces in a matrix have to be replaced by zeros). For every pair $(t, \tau) \in Z^2$ such that $k=t-\tau \geq 1$, the fundamental solution matrix is obtained from the core solution matrix $\mathbf{C}_{t,\tau}$ in eq. (4), augmented on the left by the $\boldsymbol{\varphi}_{t,\tau}$ column. That is,

$$\boldsymbol{\Phi}_{t,\tau} = (\boldsymbol{\varphi}_{t,\tau} \quad \mathbf{C}_{t,\tau}) = \begin{pmatrix} \varphi_1(\tau+1) & -1 & & & \\ \varphi_2(\tau+2) & \varphi_1(\tau+2) & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_2(t-1) & \varphi_1(t-1) & -1 \\ & & & \varphi_2(t) & \varphi_1(t) \end{pmatrix}. \tag{5}$$

Formally $\boldsymbol{\Phi}_{t,\tau}$ (since $\tau=t-k$) is a square $k \times k$ matrix whose (i,j) entry $1 \leq i, j \leq k$ is given by

$$\begin{cases} -1 & \text{if} \qquad\qquad i = j-1, \quad \text{and} \quad 2 \leq j \leq k, \\ \varphi_{1+m}(t-k+i) & \text{if} \quad m = 0,1, \quad i = j+m, \quad \text{and} \quad 1 \leq j \leq k-m, \\ 0 & \text{otherwise.} \end{cases}$$

It is a continuant or tridiagonal matrix, that is a matrix that is both an upper and lower Hessenberg matrix. We may also characterize it as a *time varying* Toeplitz matrix, because its time invariant version is a Toeplitz matrix of bandwidth 3.

For every pair $(t, \tau) \in Z^2$ with $\tau < t$, the so called principal determinant associated with eq. (5) is given by

$$\xi(t, \tau) = \det(\boldsymbol{\Phi}_{t,\tau}). \tag{6}$$

That is, $\xi(t, \tau)$ for $k \geq 2$, is a determinant of a $k \times k$ matrix; each of the two nonzero diagonals (below the superdiagonal) of this matrix consists of the time varying coefficients $\varphi_m(\cdot), m = 1, 2$ from *t-k+m* to *t*. In other words, $\xi(t,\tau)$ is a *k*-th order tridiagonal determinant. Paraskevopoulos and Karanasos (2021) give its ordinary expansion in non-determinant form (a closed form solution).

We further extend the definition of $\xi(t,\tau)$ so as to be defined over Z^2 by assigning the initial conditions:

$$\xi(t,\tau) = \begin{cases} 1 & \text{if} \quad t = \tau, \\ 0 & \text{if} \quad t < \tau. \end{cases} \tag{7}$$

2.3 Main Theorem

This short section contains the statement of our main theorem.

Theorem 1 *An equivalent explicit representation of* y_t *in eq. (1) in terms of prescribed random variables* y_τ, $y_{\tau-1}$ *is given by*

$$y_t = y_{t,k}^{hom} + y_{t,k}^{par}\,, \tag{8}$$

where

$$\begin{aligned} y_{t,k}^{hom} &= \xi(t,\tau)y_\tau + \varphi_2(\tau+1)\xi(t,\tau+1)y_{\tau-1}, \\ y_{t,k}^{par} &= \sum_{i=\tau+1}^{t} \xi(t,i)[\varphi_0(i) + \varepsilon_i] = \sum_{i=0}^{k-1} \xi(t,t-i)[\varphi_0(t-i) + \varepsilon_{t-i}]. \end{aligned}$$

In the above Theorem y_t is decomposed into two parts: first, the $y_{t,k}^{hom}$ part, which is the sum of the two fundamental solutions multiplied by observable random variables, and, second, the $y_{t,k}^{par}$ part, which is formed by products involving the principal determinant $\xi(t,i)$ multiplied by the forcing term: $\varphi_0(i) + \varepsilon_i$.

Notice that the *coefficients* of eq. (8), that is, the ξ's are expressed as continuant determinants. Moreover, for $k = 0$, that is $t = \tau$ (for $i > j$ we use the convention $\sum_{q=i}^{j}(\cdot) = 0$), since $\xi(t,t) = 1$ and $\xi(t,t+1) = 0$ (see eqs. (6) and (7)), eq. (8) becomes an identity: y_t=y_t. Similarly, when $k = 1$, that is $\tau = t-1$ eq. (8), since $\xi(t,t-1) = \varphi_1(t)$, $\xi(t,t) = 1$, reduces to $y_t = \varphi_1(t)y_{t-1} + \varphi_2(t)y_{t-2} + \varphi_0(t) + \varepsilon_t$.

The asymptotic stability problem is to provide sufficient conditions such that a class of stochastic processes solving eq. (1) approaches a solution independently of the two prescribed random variables (the effect of the prescribed random variables is gradually dying out) as $\tau \to -\infty$, that is when the homogeneous solution in eq. (8) tends to zero, under a prescribed type of convergence. The explicit representation of the homogeneous solution in eq. (8) makes it possible to provide such type of conditions in Proposition 1 ensuring the L_2 convergence to zero of the homogeneous solution, that is $y_{t,\tau}^{hom} \overset{L_2}{\to} 0$, as $\tau \to -\infty$, which means that $lim_{\tau\to-\infty}\left\|y_{t,\tau}^{hom}\right\|_{L_2} = 0$, or equivalently that $lim_{\tau\to-\infty}E(y_{t,\tau}^{hom})^2 = 0$.

Proposition 1 *If* $sup_t|\varphi_m(t)| < \infty$ *for each* m*, with* $m = 1,2$ *then a sufficient condition for an* L_2*-bounded stochastic process* y_t *(that is* $sup_t E(y_t^2) < \infty$*), which solves eq. (1) to be asymptotically stable (in* L_2 *sense) is:*

$$lim_{\tau\to\infty}\xi(t,\tau) = 0 \text{ for each } t.$$

Next, we will introduce an alternative notation for either $\mathbf{\Phi}_{t,\tau}$ or $\xi(t,\tau)$.

Notation 1 *We will use two alternative notations for the fundamental solution matrix and the principal determinant:*
i) $\mathbf{\Phi}_{t,\tau} = \mathbf{\Phi}_{t,t-k}$ *in eq. (5) can be re-expressed in an alternative notation as* $\mathbf{\Phi}_{t,k}$*, where now the second subscript denotes the order of the square matrix,*
ii) Similarly, $\xi(t,k)$ *is an alternative notation for* $\xi(t,\tau) = \xi(t,t-k)$ *in eq. (6).*

In the next Section, we illustrate the above claims in the context of a simple seasonal process with fixed periodicity, and a cyclical model as well.

3. Seasons and Cycles

3.1 Periodic AR(2) Model

Periodic regularities are phenomena occurring at the same season every year, so analogous to each other that we can view them as recurrences of the same event. Many economic time series are periodic in this sense. In the present Section we express them in a mathematical model, so that we can then employ it for forecasting and control. Gladyshev (1961) introduced a technique which still dominates the literature. He begins by decomposing the series into subperiods; then he treats each point within a subperiod as one part of a multivariate process. In this way he transforms a univariate non-stationary formulation into a multivariate stationary one. Following Gladyshev, Tiao and Grupe (1980) and Osborn (1991) treated periodic autoregressions as conventional nonperiodic VAR processes (see Appendix C for details). But, as pointed out by Lund et al. (2006), even low order specifications can have an inordinately large numbers of parameters. A PAR(1) model for daily data, for example, has 365 autoregressive parameters. Its time invariant VAR form will contain 365 variables, and this is a handicap, especially for forecasting.

To simplify our exposition, we also introduce the following notation for the seasonal model:
Notation 2 $T \in Z_{>0}$ *denotes the periods (i.e., years);* $s = 1, \dots, l$*, denotes the seasons (i.e, quarters in a year:* $l = 4$*),* $l \in Z_{>0}$*.*
Now time is represented by $t \stackrel{\text{def}}{=} t_s$*, where* $t_s = (T-1)l + s$*. That is, time* t_s *(or for notational ease t) is at the* s*-th season of period* T*.*

The most common case is the modeling in one dimensional time repetition at equal intervals. In this Section we present a re-examination of the periodic modeling problem. Our approach differs from most of the existing literature in that we stay within the univariate framework (see also Karanasos et al., 2014a).

A periodic AR model of order 2 with l seasons, PAR(2; l), can be expressed as the TV-AR(2) model in eq. (1):

$$y_t = \varphi_0(t) + \varphi_1(t)y_{t-1} + \varphi_2(t)y_{t-2} + \varepsilon_t, \tag{9}$$

where $t \stackrel{\text{def}}{=} t_s$, that is time t_s is at the s-th season and the periodically (or seasonally) varying coefficients $\varphi_m(t)$, $m = 1,2$ are constant in each season:

$$\varphi_m(t_s) \stackrel{\text{def}}{=} \varphi_{ms}. \tag{10}$$

For example, if $s = 1$ (that is, we are at the l-th season) then the periodically varying parameters are φ_{ml} whereas if $s = 1$ (that is, we are at the 1st season) then the periodically varying parameters are φ_{m1}. The above process nests the AR(2) model as a special case if we assume that the drift and all the AR parameters are constant, that is: $\varphi_{ms} = \varphi_m$, $m = 0,1,2$ for all s.

In what follows we also assume, for ease of presentation and without loss of generality, that $k = nl$, $n \in Z_{>0}$, that is $\tau = t - nl$ (it can of course be given at any time $\tau = t - (n-1)l + s$).

Since for the periodic process, $t \stackrel{\text{def}}{=} t_s = (T-1)l + s$, and $k = nl$, we will make use of the following alternative notation.

Notation 3 *i)* $\mathbf{\Phi}_{t,nl}$ *(see Notation 1(i)) or* $\mathbf{\Phi}_{t_s,nl}$ *can be re-expressed in an alternative notation as* $\mathbf{\Phi}_{s,nl}$,
ii) Similarly, $\xi(s,nl)$ *(see Notation 1(ii)) is an alternative notation for* $\xi(t_s,nl)$.

For the PAR(2;l) model the continuant matrix $\mathbf{\Phi}_{t,\tau}$ in eq. (5) or $\mathbf{\Phi}_{s,nl}$ can be expressed as a block Toeplitz matrix. Thus, we have

$$\xi(s,nl) = |\mathbf{\Phi}_{s,nl}|, \tag{11}$$

with

$$\mathbf{\Phi}_{s,nl} = \begin{pmatrix} \mathbf{\Phi}_{s,l} & \mathbf{0}_l & & & \\ \mathbf{0}_{s,l} & \mathbf{\Phi}_{s,l} & \mathbf{0}_l & & \\ & \ddots & \ddots & \ddots & \\ & & \mathbf{0}_{s,l} & \mathbf{\Phi}_{s,l} & \mathbf{0}_l \\ & & & \mathbf{0}_{s,l} & \mathbf{\Phi}_{s,l} \end{pmatrix}, \tag{12}$$

where $\mathbf{0}_l$ is an $l \times l$ matrix of zeros except for -1 in its (l,1)-th entry; $\mathbf{0}_{s,l}$ is an $l \times l$ matrix of zeros except $\varphi_{2,s+1}$, in its (1,l)-th entry and the block diagonal matrix $\mathbf{\Phi}_{s,l}$ is the continuant or tridiagonal matrix given by

$$\mathbf{\Phi}_{s,l} = \begin{pmatrix} \varphi_{1,s-l+1} & -1 & & & \\ \varphi_{2,s-l+2} & \varphi_{1,s-l+2} & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_{2,s} & \varphi_{1,s-1} & -1 \\ & & & \varphi_{2,s} & \varphi_{1,s} \end{pmatrix}, \tag{13}$$

where for $s \leq$ j we replace $s - j$ by $s - j + l$ (we recall that $\varphi_m(t_s) \overset{\text{def}}{=} \varphi_{ms}$, see eq. (10)). For example, if either $s = l$ or $s = 1$, then

$$\mathbf{\Phi}_{l,l} = \begin{pmatrix} \varphi_{11} & -1 & & & \\ \varphi_{22} & \varphi_{12} & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_{2l} & \varphi_{1,l-1} & -1 \\ & & & \varphi_{2l} & \varphi_{1l} \end{pmatrix}, \mathbf{\Phi}_{1,l} = \begin{pmatrix} \varphi_{12} & -1 & & & \\ \varphi_{23} & \varphi_{13} & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_{2l} & \varphi_{1l} & -1 \\ & & & \varphi_{21} & \varphi_{11} \end{pmatrix}.$$

Next we will make use of the above block Toeplitz matrix to obtain an explicit formula of $\xi_{s,nl}$ in which we decompose it into tridiagonal determinants, $\xi_{s,l}$. To prepare the reader, before we present the main result we consider the case where $n = 2$ that is we go from time t back to time $t - 2l$. The tridiagonal determinant $\xi_{s,2l}$ can be written as the sum of two terms

$$\xi(s, 2l) = \begin{vmatrix} \mathbf{\Phi}_{s,l} & \mathbf{0}_l \\ \mathbf{0}_{s,l} & \mathbf{\Phi}_{s,l} \end{vmatrix} = \xi^2(s, l) + \varphi_{2,s+1}\xi(s, l-1)\xi(s-1, l-1), \tag{14}$$

where each term is the product of two continuant (or tridiagonal) determinants.

Next let $i_j \in \{0,1\}$, $j = 1,...,n-1$, and define

$$\phi_{j,s} \overset{\text{def}}{=} \phi_j \quad : \quad \phi_j = \begin{cases} 1 & \text{if} \quad i_j = 0, \\ \varphi_{2,s+1} & \text{if} \quad i_j = 1. \end{cases} \tag{15}$$

Let also $\prod_{g=i}^{j}(\cdot) = 1$ for $i < j$. We recall that $\xi(s, nl)$ in eq. (11), is the determinant of $\mathbf{\Phi}_{s,nl}$ in eq. (12).

Proposition 2 *For the PAR(*$2; l$*) process,* $\xi(s, nl)$, *for* $n \geq 2$, *can be written as*

$$\xi(s, nl) = \sum_{i_1=0}^{1} \cdots \sum_{i_{n-1}=0}^{1} \{\xi(s, l - i_1)[\prod_{g=2}^{n-1} \phi_{g-1}\xi(s - i_{g-1}, l - i_g - i_{g-1})]\phi_{n-1}\xi(s - i_{n-1}, l - i_{n-1})\}, \tag{16}$$

where $\xi(s, nl) = |\mathbf{\Phi}_{s,l}|$, $\mathbf{\Phi}_{s,l}$ *is given by eq. (13) and* ϕ_j *is defined in eq. (15).*

In the above Proposition (its proof is presented in Appendix B) $\xi(s, nl)$ is expressed as the sum of $\sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}$ terms each of which is the product of n terms. In other words, it is decomposed into determinants of continuant matrices, $m = 0,1,2$: $\mathbf{\Phi}_{s-i_{g-1}, l-i_g-i_{g-1}}$.

When $n = 3$ eq. (16) reduces to:

$$\xi(s,3l) = \xi^3(s,l) + \varphi_{2,s+1}[\xi(s,l-1)\xi(s-1,l-1)\xi(s,l) + \xi(s,l)\xi(s,l-1)\xi(s-1,l-1) + \varphi_{2,s+1}\xi(s,l-1)\xi(s-1,l-2)\xi(s-1,l-1)]$$
$$= \xi^3(s,l) + \varphi_{2,s+1}\xi(s,l-1)\xi(s-1,l-1)[2\xi(s,l) + \varphi_{2,s+1}\xi(s-1,l-2)],$$

that is, $\xi(s,nl)$ is equal to the sum of four ($p^{n-1} = 2^2$; $i_1 = i_2 = 0$; $i_1 = i_2 = 1$; $i_1 = 0$ *and* $i_2 = 1$; $i_1 = 1$ *and* $i_2 = 0$) terms each of which is the product of three ($n = 3$) ξ's (continuant determinants).

When $n = 2$, eq. (16) reduces to eq. (14):

$$\xi(s,2l) = \sum_{i_1=0}^{1}\xi(s,l-i_1)\phi_1\xi(s-i_1,l-i_1) = \underbrace{\xi^2(s,l)}_{i_1=0} + \underbrace{\varphi_{2,s+1}\xi(s,l-1)\xi(s-1,l-1)}_{i_1=1}.$$

Proposition 3 *An equivalent explicit representation of y_t in eq. (9) in terms of the two prescribed random variables y_{t-nl}, y_{t-nl-1} is given by*

$$y_t = y_{t,nl}^{hom} + y_{t,nl}^{par}, \tag{17}$$

where

$$y_{t,nl}^{hom} = \xi(s,nl)y_{t-nl} + \varphi_{2,s+1}\xi(s,nl-1)y_{t-nl-1},$$

$$y_{t,nl}^{par} = \sum_{i=0}^{l-1}\sum_{j=0}^{n-1}\xi(s,i+jl)\varphi_{0,s-i} + \sum_{j=0}^{nl-1}\xi(s,j)\varepsilon_{t-j},$$

and $\xi(s,nl)$ is given either in eq. (11) or in Proposition (2).

The proof of eq. (17) in the above Proposition follows immediately from Theorem 1 and the definition of the periodic model (9).

3.2 Cyclical AR(2) Process

Some economic series exhibit oscillations which are not associated with the same fixed period every year. Despite their lack of fixed periodicity, such time series are predictable to a certain degree.

Rather than setting up a general model from first principles, we re-interpret the periodic model with some modifications.

Before proceeding further, some additional notation is required.

Notation 4 *We assume that we have* d *cycles, with* $1 \le d \le l$.
Then $s_j = l_{j-1} + 1, \dots, l_j$ *j=1,…,d,(with* $0 = l_0 < l_1 < \cdots < l_d = l$*) are the seasons in cycle* j*. Thus we can write* $t \stackrel{\text{def}}{=} t_{s_j} = (T-1)l + s_j$.

A CAR(2) model with l seasons and d cycles (CAR(2;*l*;*d*)) is defined as a TV-

AR(2) model:

$$y_t = \varphi_0(t) + \varphi_1(t)y_{t-1} + \varphi_2(t)y_{t-2} + \varepsilon_t, \tag{18}$$

where $\varphi_m(t) \stackrel{\text{def}}{=} \varphi_m(t_{s_j})$ is given by

$$\varphi_m\left(t_{s_j}\right) \stackrel{\text{def}}{=} \varphi_{m,s_j}, m = 0,1,2. \tag{19}$$

In what follows, for notational ease and without loss of generality we will assume that current time *t*, is at the last season of the last cycle, that is $j = d$ and $s_d = l_d = l: t \stackrel{\text{def}}{=} t_l = Tl.$

For the above process, $\Phi_{l,l}$ in eq. (13) can be written as

$$\mathbf{\Phi}_{l,l} = \begin{bmatrix} \mathbf{\Phi}_{l_1,l_1} & \mathbf{0}_{d-1} & & & \\ \overline{\mathbf{0}}_{d-1} & \mathbf{\Phi}_{l_2,l_2-l_1} & \mathbf{0}_{d-2} & & \\ & \ddots & \ddots & \ddots & \\ & & \overline{\mathbf{0}}_2 & \mathbf{\Phi}_{l_{d-1},l_{d-1}-l_{d-2}} & \mathbf{0}_1 \\ & & & \overline{\mathbf{0}}_1 & \mathbf{\Phi}_{l\ \ ,l\ \ -l_{d-1}} \end{bmatrix} \tag{20}$$

where i) the *j*-th ($j = 1,\ldots,d$) block of the main diagonal is $\mathbf{\Phi}_{l_j,l_j-l_{j-1}}$, which is a $(l_j - l_{j-1}) \times (l_j - l_{j-1})$ banded 'time varying' Toeplitz matrix of bandwidth 3:

$$\mathbf{\Phi}_{l_j,l_j-l_{j-1}} = \begin{pmatrix} \varphi_{1,l_{j-1}+1} & -1 & & & \\ \varphi_{2,l_{j-1}+2} & \varphi_{1,l_{j-1}+2} & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_{2,l_j-1} & \varphi_{1,l_j-1} & -1 \\ & & & \varphi_{2,l_j} & \varphi_{1,l_j} \end{pmatrix},$$

ii) the *j*-th ($\mathrm{j} = 1,\ldots,d\text{-}1$) block in the subdiagonal, is a $(l_j - l_{j-1}) \times (l_{j+1} - l_j)$ matrix of zeros except for $\varphi_{2,l_{d-j}+1}$ in its $1 \times (l_{j+1} - l_j)$ entry, and iii) the 0_j block in the superdiagonal, is a $(l_{j+1} - l_j) \times (l_j - l_{j-1})$ matrix of zeros except for -1 in its $(l_{j+1} - l_j) \times 1$ entry, and iv) there are zeros elsewhere.

Next we define $\varphi_{j,l_j} \stackrel{\text{def}}{=} \varphi_j$:

$$\phi_j = \begin{cases} 1 & \text{if} \quad i_j = 0, \\ \varphi_{2,l_j+1} & \text{if} \quad i_j = 1. \end{cases}$$

We recall that $\xi(l,l) = \left|\mathbf{\Phi}_{l,l}\right|$, that is $\xi(l,l)$ is the determinant of $\mathbf{\Phi}_{l,l}$ in eq. (20).

Proposition 4 *For the CAR(2;l;d) process in eqs. (18) and (19), with $2 \le d \le l$, $\xi(l,l)$ can be written as*

$$\xi(l,l) = \sum_{i_1=0}^{1} \cdots \sum_{i_{d-1}=0}^{1} \{\xi(l, l - l_{d-1} - i_1)\left[\prod_{g=2}^{d-1} \phi_{2,d-g+1}\xi(l_{d-g+1} - i_{g-1}, l_{d-g+1} - l_{d-g} - i_g - i_{g-1})\right] \phi_{2,d-1}\xi(l_1 - i_{d-1}, l_1 - i_{d-1})\}. \quad (21)$$

(the proof of Proposition 4 is similar to that of Proposition 2).

For example, when we have two cycles, that is $d = 2, \xi(l,l)$ in eq. (21) is reduced to:

$$\xi(l,l) = \sum_{i_1=0}^{1} \xi(l, l - l_1 - i_1)\phi_{2,1}\xi(l_1 - i_1, l_1 - i_1) =$$

$$\underbrace{\xi(l, l - l_1)\xi(l_1, l_1)}_{i_1=0} + \underbrace{\xi(l, l - l_1 - 1)\varphi_{2,l_1+1}\xi(l_1 - 1, l_1 - 1)}_{i_1=1}.$$

4. Abrupt Breaks

Our general result has been presented in Section 2.3. In the current Section, we discuss still another example in order to both make our analysis clearer and to demonstrate its applicability. One important case is that of r, $0 \le r \le k - 1$, abrupt breaks at times t-k_1, t-k_2, ..., t-k_r, where $0 = k_0 < k_1 < k_2 < \cdots < k_r < k_{r+1} = k$, $k_r \in Z_{>0}$. That is, between $t - k = t - k_{r+1}$ and the present time $t = t - k_0$ the AR(2) process contains r structural breaks and the switch from one set of parameters to another is abrupt. In particular

$$y_\tau = \varphi_{0j} + \varphi_{1j} y_{\tau-1} + \varphi_{2j} y_{\tau-2} + \sigma_j^2 e_{\tau,j}, \quad (22)$$

For $\tau = t - k_{j-1}, \ldots, t - k_j + 1, j = 1, \ldots, r + 1$ and $e_{t,j}$ i.i.d (0,1) $\forall$ *t,j*. Within the class of AR(2) processes, this specification is quite general and allows for intercept and slope shifts as well as changes in the error variances (see also Pesaran et al., 2006). Each regime *j* is characterized by φ_{0j}, a vector of autoregressive coefficients: φ_j, and an error term variance, $0 < \sigma_j^2 < M_j \ \forall \ j$, $M_j \in Z_{>0}$. We term this model abrupt breaks AR process of order (2;*r*): ABAR(2;*r*).

For the AR(2) model with *r* abrupt breaks, $\xi(t,\tau)$, $\tau = t - k$, in eq. (6) can be written as the determinant of a partitioned (or a block) tridiagonal matrix

$$\xi(t,\tau) = \begin{vmatrix} \mathbf{\Phi}_{t-k_r,k_{r+1}-k_r} & \mathbf{0}_r & & & \\ \overline{\mathbf{0}}_r & \mathbf{\Phi}_{t-k_{r-1},k_r-k_{r-1}} & \mathbf{0}_{r-1} & & \\ & \ddots & \ddots & \ddots & \\ & & \overline{\mathbf{0}}_2 & \mathbf{\Phi}_{t-k_1,k_2-k_1} & \mathbf{0}_1 \\ & & & \overline{\mathbf{0}}_1 & \mathbf{\Phi}_{t,k_1} \end{vmatrix}, \quad (23)$$

where first, the *j*-th (*j=1,...r+1)* block of the main diagonal is $\mathbf{\Phi}_{t-k_{j-1},k_j-k_{j-1}}$, which is a $(k_j - k_{j-1}) \times (k_j - k_{j-1})$ banded Toeplitz matrix of bandwidth 3:

$$\mathbf{\Phi}_{t-k_{j-1},k_j-k_{j-1}} = \begin{pmatrix} \varphi_{1j} & -1 & & & \\ \varphi_{2j} & \varphi_{1j} & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_{2j} & \varphi_{1j} & -1 \\ & & & \varphi_{2j} & \varphi_{1j} \end{pmatrix}, j$$

with

$\xi(t-k_{j-1}, k_j-k_{j-1}) = \left|\mathbf{\Phi}_{t-k_{j-1},k_j-k_{j-1}}\right| = \frac{1}{\lambda_{1j}-\lambda_{2j}} = (\lambda_{1j}^{k_j-k_{j-1}+1} - \lambda_{2j}^{k_j-k_{j-1}+1})$, and the second equality holds if and only if $\lambda_{1j} \neq \lambda_{2j}$ (where $1-\varphi_{1j}B-\varphi_{2j}B^2 = (1-\lambda_{1j}B)(1-\lambda_{2j}B)$; second, the *j*-th (j = 1,...,*r*) block of the subdiagonal, is a $(k_j - k_{j-1}) \times (k_{j+1}-k_j)$ matrix of zeros except for φ_{2j} in its $1 \times (k_{j+1}-k_j)$ entry; and third, the *j*-th block of the superdiagonal, is a $(k_{j+1}\text{-}k_j) \times (k_j - k_{j-1})$matrix of zeros except for -1 in its $(k_{j+1}-k_j) \times 1$ entry, and iv) there are zeros elsewhere.

Next we define ϕ_j (j = 1,...,*r*):

$$\phi_j = \begin{cases} 1 & \text{if} \quad i_j = 0, \\ \varphi_{2j} & \text{if} \quad i_j = 1. \end{cases}$$

We also recall the Notation 1(ii), that is $\xi(t, t-k) \overset{\text{def}}{=} \xi(t,k)$.

Proposition 5 *For the ABAR (*$2; r$*) process in eq. (22),* $\xi(t,k)$ *can be written as*

$$\xi(t,k) = \sum_{i_1=0}^{1} \cdots \sum_{i_r=0}^{1} \{\xi(t, k_1 - i_1)[\prod_{g=2}^{r} \phi_{2,g-1} \xi(t-k_{g-1}-i_{g-1}, k_g - k_{g-1} - i_g - i_{g-1})] \phi_{2r} \xi(t-k_r-i_r, k-k_r-i_r)\} \quad (24)$$

(the proof of Proposition 7 is similar to that of Proposition 2).

As an example consider the case with one break, that is $r = 1$. Then $\xi(t,k)$ in eq. (24) reduces to:

$$\xi(t,k) = \sum_{i_1=0}^{1} \xi(t, k_1 - i_1) \phi_{21} \xi(t-k_1-i_1, k-k_1-i_1)$$
$$= \underbrace{\xi(t,k_1)\xi(t-k_1, k-k_1)}_{i_1=0} + \underbrace{\xi(t,k_1-1)\varphi_{21}\xi(t-k_1-1, k-k_1-1)}_{i_1=1}.$$

Proposition 6 *An equivalent explicit representation of* y_t *in eq. (22) in terms of the two prescribed random variables* y_{t-k}, y_{t-k-1} *, is given by*

$$y_t = y_{t,k}^{hom} + y_{t,k}^{par},$$

where

$$y_{t,k}^{hom} = \xi(t,k) y_{t-k} + \varphi_{2,r+1} \xi(t, k-1) y_{t-k-1},$$
$$y_{t,k}^{par} = \sum_{j=1}^{r+1} \left[\varphi_{0,j} \sum_{i=k_{j-1}}^{k_j - 1} \xi(t,i) + \sigma_j^2 \sum_{i=k_{j-1}}^{k_j - 1} \xi(t,i) e_{t-i,j} \right],$$

and $\xi(t,k)$ *is given either in eq. (23) or in Proposition 5.*

The proof of the above Proposition follows immediately from Theorem 1 and the definition of the ABAR(2; r) model in eq. (22).

5. Prediction and Moment Structure

We turn our attention to the fundamental properties of the various TV-AR(2) processes. Armed with a powerful technique for manipulating time varying models we may now provide a thorough description of the processes (1) by deriving, first, its multistep ahead predictor, the associated forecast error and the mean square error; second, the first two unconditional moments of this process, and third, its covariance structure.

5.1 Multi Step Forecasts

Forecasts Based on Finite Observations. In what follows we discuss an approach focusing on small size sample forecasts. In this case the optimal linear forecast is based on the two past observations, including a nonzero time varying deterministic drift $\varphi_0(t)$. Let K_τ ($\tau = t - k$) be the subspace of L_2 spanned by the set of two past observations $\{1, y_\tau, y_{\tau-1}\}$ (also containing all constant functions). Following Karanasos *et al.* (2022), we shall denote the orthogonal projection of y_t onto K_τ by $\hat{E}(y_t|K_\tau)$ and we shall refer to it as the optimal linear predictor of y_t, based on K_τ.

The approach employed herein takes advantage of the explicit form of the process y_t, established by eq. (8). Assuming further that $\{\varepsilon_t\}$ is a martingale difference sequence, relative to y_τ, then applying the optimal linear predictor of y_t in eq. (8), based on K_τ, the next Proposition follows:

Proposition 7 *For the TV-AR(2) model the* k*-step-ahead optimal (in* L_2*-sense) linear predictor of* y_t *based on* K_τ*, is readily seen to be*

$$\hat{E}(y_t|K_\tau) = \sum_{i=0}^{k-1} \xi(t,t-i)\varphi_0(t-i) + \xi(t,\tau)y_\tau + \varphi_2(\tau+1)\xi(t,\tau+1)\, y_{\tau-1}. \tag{25}$$

The forecast error for the k-step-ahead predictor, and its associated mean square error (its variance), are given by:

$$FE_{t,\tau} = \textstyle\sum_{i=0}^{k-1} \xi(t,t-i)\varepsilon_{t-i}, \tag{26}$$

$$MSE_{t,\tau} = \textstyle\sum_{i=0}^{k-1} \xi^2(t,t-i)\sigma^2(t-i). \tag{27}$$

The following Proposition presents results for the forecasts from PAR and ABAR processes. We recall Notation 1(ii): $\xi(t,t-k) \stackrel{\text{def}}{=} \xi(t,k)$.

Proposition 8 *For the PAR(2;l) model the* nl*-step-ahead linear predictor in eq. (25) it becomes*

$$\hat{E}(y_t|K_\tau) = \sum_{i=0}^{l-1}\sum_{j=0}^{n-1}\xi(s,i+jl)\varphi_{0,s-i} + \xi(s,nl)y_{t-nl} + \varphi_{2,s+1}\xi(s,nl-1)\,y_{t-nl-1}.$$

where $\xi(s,nl)$ *is given in Proposition (2).*

For the ABAR($2;r$*) model in eq. (22) the* k*-step-ahead optimal linear predictor is given by*

$$\hat{E}(y_t|K_\tau) = \sum_{j=1}^{r+1}\varphi_{0,j}\sum_{i=k_{j-1}}^{k_j-1}\xi(t,t-i) + \xi(t,k)y_{t-k} + \varphi_{2,r+1}\xi(t,k-1)\,y_{t-k-1}.$$

Where $\xi(t,i)$ *is given either in eq. (23) or in Proposition 5.*

Franses and Paap (2005) employ the vector season representation to compute forecasts and forecast error variances for a PAR(1;4) process. In this way forecasts can be generated along the same lines with quadrivariate VAR(1) models. Franses (1996a) derives multi-step forecast error variances for low-order PAR models with $l = 4$, using the VS representation. But, if l is large even low order specifications will have large VAR representations and this is a handicap especially for forecasting. In contrast, our formulae using the univariate framework allow a fast computation of the multi-step-ahead predictors even if l is large.

In what follows we give conditions for the first and second unconditional moments of model (1) to exist.

5.2 Wold-Cramér Representation

In Proposition 9, we provide the existence of the Wold-Cramér decomposition (see Cramér, 1961)[4] and, therefore, impulse response functions (IRFs), for the model in eq. (1).

First, we need a Condition.

Condition 1 $\sum_{i=0}^{\infty}|\xi(t,t-i)| < \infty$ (absolute summability condition).

Condition 1 which, along with the boundedness of the drift, ensure the existence of the Wold-Cramér decomposition (see Proposition 9) of TV-AR(2) processes, which is second order, that is of finite first two unconditional moments and autocovariance function (see Propositions 10-12).

Proposition 9 *Let the absolute summability condition hold. Let also* $\varphi_0(t)$ *be bounded function in* t*. Then there exists a solution of eq. (1) in* L_2 *of the form:*

[4]Since a non-stationary generalization of Wold's result was given by Cramér, it is referred to as Wold-Cramér decomposition.

$$y_t = \sum_{i=0}^{\infty} \xi(t, t-i)[\varphi_0(t-i) + \varepsilon_{t-i}]. \tag{28}$$

A direct proof of of Proposition 9 is available upon request. The solution of eq. (1) in eq. (28) is decomposed into two orthogonal parts, a deterministic part and a mean zero random part, that is, E(y_t)= $\sum_{i=0}^{\infty} \xi(t, t-i)\varphi_0(t-i)$ is the non random part (see eq. 29) in Proposition 10 below), while $lim_{k\to\infty} FE_{t,\tau} = \sum_{i=0}^{\infty} \xi(t, t-i)\varepsilon_{t-i}$, (i.e., the limit of forecast errors, see eq. (26) is the mean zero random part of y_t. As eq. (28) is future independent, we shall also referred to it as a causal solution of TV-AR(2) models.

Another immediate consequence of Theorem 1 is the following Proposition (its proof is available upon request), where we state an expression for the first unconditional moment of y_t.

Proposition 10 *Let the conditions of Proposition 9 hold. Then the unconditional mean of the process in eq. (1), exists in* **R** *and is given by*

$$E(y_t) = lim_{k\to\propto} E(y_t | F_\tau) = \sum_{i=0}^{\infty} \xi(t, t-i)\varphi(t-i). \tag{29}$$

5.3 Second Moments

In this subsection we state as a Proposition the result for the second moment structure.

Proposition 11 *Let Condition 1 hold. Then the unconditional variance of* y_t *in eq. (1) is given by*

$$Var(y_t) = \sum_{i=0}^{\infty} \xi^2(t, t-i)\sigma^2(t-i). \tag{30}$$

Necessary conditions for the y_t *process to be first and second order respectively are:*

$$lim_{\tau\to-\infty}\xi(t,\tau)\varphi(\tau)=0 \text{ and } lim_{\tau\to-\infty}\xi^2(t,\tau)\sigma^2(\tau) = 0 \text{ for all } t.$$

Moreover, the stability condition, that is $lim_{\tau\to-\infty}\xi(t,\tau)=0$*, is sufficient for the above two limits to exist, due to the boundedness of* $\varphi(r)$ *and* $\sigma^2(r)$*, while it is necessary for the absolute summability to hold.*

Notice that the unconditional variance is the limit of the MSE in eq. (27) as $k \to \infty$.

The main logical connections between the conditions, described in the above Proposition, are summarized in the following commutative diagrams (the proof of (31) is available upon request):

$$\begin{array}{ccccc}
\sum_{r=-\infty}^{t}\xi(t,r)\varphi(r)\in R & \Leftarrow & \sum_{r=-\infty}^{t}|\xi(t,r)|<\infty & \Rightarrow & \sum_{r=-\infty}^{t}\xi^2(t,r)\sigma^2(r)<\infty \\
\Downarrow & \swarrow & \Downarrow & \searrow & \Downarrow \\
lim_{\tau\to-\infty}\xi(t,\tau)\varphi(\tau)=0 & \Leftarrow & lim_{\tau\to-\infty}\xi(t,\tau)=0 & \Rightarrow & lim_{\tau\to-\infty}\xi^2(t,\tau)\sigma^2(\tau)=0.
\end{array}$$

Figure 1: Commutative Diagrams

(31)

In the following Proposition, we state an explicit expression for the covariance structure for the Wold-Cramér solution decomposition of the TV-AR(2) process.

Proposition 12 *Let the conditions of Proposition 9 hold. Then time varying ℓ -th order autocovariance function,* $\gamma_t(\ell)$=Cov$(y_t, y_{t-\ell})$, $\ell \in Z_{\geq 0}$, *of* y_t *in eq. (1), exists in* R *and is given by*

$$\gamma_t(\ell) = \sum_{r=-\infty}^{t-\ell} \xi(t,r)\xi(t-\ell,r)\sigma^2(r),$$
$$(\text{for } \ell \geq 1) = \xi(t,\ell)Var(y_{t-\ell}) + \varphi_2(t-\ell+1)\xi(t,\ell-1)\gamma_{t-\ell}(1).$$

(32)

The time-varying variance of y_t in eq. (30), is recovered by applying $\gamma_t(\ell)$ in eq. (32) for $\ell = 0$ that is $\gamma_t(0)$=Var(y_t). Moreover, the absolute summability condition implies absolute summable autocovariances: $\sum_{\ell=0}^{\infty}|\gamma_t(\ell)| < \infty$ for all *t* (formal proofs of Propositions 11 and 12 are not presented but are available upon request).

6. Conclusions

We have provided the general solutions to low order TV-AR models in terms of their homogeneous and particular parts. Our first step was to find the fundamental set of solutions by computing the determinants of the matrix of coefficients associated with the infinite linear system that represents the difference equation.

The framework developed in Section 2, proved itself to be a general time varying theory, encompassing a number of seemingly unrelated models, discussed in Sections 3 and 4. We have identified common properties (throughout the paper and in particular in Section 5), which are basic to each of the particular application.

We believe that time varying models should take center stage in the time series literature; this is why we have labored to develop a theory with rigorous foundations that can encompass a variety of dynamic systems, i.e., periodic and cyclical processes, and AR models which contain multiple structural breaks. Work that remains to be done by us and fellow researchers is on estimation and testing (for one application on this front see the papers by Karanasos *et al*., 2014 and Canepa *et al*., 2022) to demonstrate the usefulness of time varying models. In the long run, a sound mathematical theory has to be cointegrated with its applicability.

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A APPENDIX

In this appendix we prove Theorem 1. Before proceeding with the main body of the proof, we present two essential tools for carrying it out.

The Infinite Gaussian Elimination. Following Paraskevopoulos (2012), we apply the infinite Gaussian elimination algorithm implemented under a rightmost pivot strategy to the coefficient matrix of eq. (3).

The process is briefly described below. We recall that $\tau = t - k$.
Call $\mathbf{H}^{(1)} = (-\varphi_2(\tau+1) \quad -\varphi_1(\tau+1) \quad 1 \quad 0 \quad \cdots)$ the opposite-sign first row of $\mathbf{\Phi}$. Insert the second row of $\mathbf{\Phi}$ below $\mathbf{H}^{(1)}$ to build the matrix $\mathbf{B}^{(2)}$:

$$\mathbf{B}^{(2)} = \begin{pmatrix} -\varphi_2(\tau+1) & -\varphi_1(\tau+1) & 1 & 0 & \dots \\ 0 & \varphi_2(\tau+2) & \varphi_1(\tau+2) & -1 & \dots \end{pmatrix}.$$

Use as pivot the rightmost one of $\mathbf{H}^{(1)}$ to clear the element $\varphi_1(\tau+2)$ in the second row of $\mathbf{B}^{(2)}$. After normalization it yields the matrix:

$$\mathbf{H}^{(2)} = \begin{pmatrix} -\varphi_2(\tau+1) & -\varphi_1(\tau+1) & 1 & 0 & \dots \\ -\varphi_2(\tau+1)\varphi_1(\tau+2) & -\varphi_2(\tau+2) - \varphi_1(\tau+1)\varphi_1(\tau+2) & 0 & 1 & \dots \end{pmatrix}.$$

Insert the third row of $\mathbf{\Phi}$ below $\mathbf{H}^{(2)}$ to build the matrix $\mathbf{B}^{(3)}$:

$$\begin{pmatrix} -\varphi_2(\tau+1) & -\varphi_1(\tau+1) & 1 & 0 & 0 & \dots \\ -\varphi_2(\tau+1)\varphi_1(\tau+2) & -\varphi_2(\tau+2)-\varphi_1(\tau+1)\varphi_1(\tau+2) & 0 & 1 & 0 & \dots \\ 0 & 0 & \varphi_2(\tau+3) & \varphi_1(\tau+3) & -1 & \dots \end{pmatrix}.$$

Use the first two rows of $\mathbf{B}^{(3)}$ as pivot rows and their rightmost 1 s as pivot elements to clear the entries $\varphi_2(\tau+3)$ and $\varphi_1(\tau+3)$ of $\mathbf{B}^{(3)}$, producing the matrix $\mathbf{H}^{(3)}$:

$$\mathbf{H}^{(3)} = \begin{pmatrix} h_{11} & h_{12} & 1 & 0 & 0 & 0 & \dots \\ h_{21} & h_{22} & 0 & 1 & 0 & 0 & \dots \\ h_{31} & h_{32} & 0 & 0 & 1 & 0 & \dots \end{pmatrix},$$

where the entries of the first column of $\mathbf{H}^{(3)}$ are given by

$$h_{11} = -\varphi_2(\tau+1),\ h_{21} = -\varphi_2(\tau+1)\varphi_1(\tau+2),$$
$$h_{31} = -\varphi_2(\tau+1)\varphi_1(\tau+2)\varphi_1(\tau+3) - \varphi_2(\tau+1)\varphi_2(\tau+3),\dots$$

and the entries of the second column are given by

$$h_{12} = -\varphi_1(\tau+1),\ h_{22} = -\varphi_2(\tau+2) - \varphi_1(\tau+1)\varphi_1(\tau+2),$$
$$h_{32} = -\varphi_1(\tau+1)\varphi_1(\tau+2)\varphi_1(\tau+3) - \varphi_2(\tau+2)\varphi_1(\tau+3) - \varphi_2(\tau+3)\varphi_1(\tau+1).$$

This process continues ad infinitum, generating an infinite chain of submatrices

$$\mathbf{H}^{(1)} \sqsubset \mathbf{H}^{(2)} \sqsubset \mathbf{H}^{(3)} \sqsubset \dots \sqsubset \mathbf{H}$$

whose limit row-finite matrix $\mathbf{H}$ is the Hermit Form (HF) of $\mathbf{\Phi}$. The i-th row of $\mathbf{H}$ is defined to be the last row of $\mathbf{H}^{(i)}$.

Two Fundamental Solutions. The opposite-sign two first columns of $\mathbf{H}$ augmented at the top by (1,0) and (0,1), respectively, that is

$\xi_\tau^{(2)} = (1,0,\ \varphi_2(\tau+1),\ \varphi_2(\tau+1)\varphi_1(\tau+2),\ \varphi_2(\tau+1)\varphi_1(t+2)\varphi_1(\tau+3) + \varphi_2(\tau+1)\varphi_2(t+3),\dots)'$,

$\xi_\tau^{(1)} = (1,0,\ \varphi_1(\tau+1),\ \varphi_2(\tau+2)+\varphi_1(t+1)\varphi_1(\tau+2),\ \varphi_1(\tau+1)\varphi_1(t+2)\ \varphi_1(\tau+3)+\varphi_2(t+2)\varphi_1(\tau+3) + \varphi_2(t+3)\varphi_1(\tau+1),\dots)'$

are the two linearly independent solution sequences of the space of homogeneous solutions of eq. (2). The linear independence of $\xi_\tau^{(1)}, \xi_\tau^{(2)}$ follows from the fact that they possess the Casoratian:

$$det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0.$$

We observe that the terms of the sequences of the two ξ's are expansions of the following determinants

$$\xi_\tau^{(2)} = \begin{cases} 1 \\ 0 \\ \varphi_2(\tau+1) \\ det\begin{pmatrix} \varphi_2(\tau+1) & -1 \\ 0 & \varphi_1(\tau+2) \end{pmatrix} \\ det\begin{pmatrix} \varphi_2(\tau+1) & -1 & 0 \\ 0 & \varphi_1(\tau+2) & -1 \\ 0 & \varphi_2(\tau+3) & \varphi_1(\tau+3) \end{pmatrix} \\ \vdots \end{cases}, \tag{A.1}$$

$$\xi_\tau^{(1)} = \begin{cases} 0 \\ 1 \\ \varphi_1(\tau+1) \\ det\begin{pmatrix} \varphi_1(\tau+1) & -1 \\ \varphi_2(\tau+2) & \varphi_1(\tau+2) \end{pmatrix} \\ det\begin{pmatrix} \varphi_1(\tau+1) & -1 & 0 \\ \varphi_2(\tau+2) & \varphi_1(\tau+2) & -1 \\ 0 & \varphi_2(\tau+3) & \varphi_1(\tau+3) \end{pmatrix} \\ \vdots \end{cases}. \tag{A.2}$$

The first few terms of the homogeneous solution sequences, as shown above, suggest that the general terms of the two ξ's (in eqs. (A.1) and (A.2), respectively) are

$$\xi^{(m)}(t,k) = det(\mathbf{\Phi}_{t,k}^{(m)}), \quad m = 1{,}2 \tag{A.3}$$

(we recall Notation 1(i): $\mathbf{\Phi}_{t,t-k} \stackrel{\text{def}}{=} \mathbf{\Phi}_{t,k}$, and (ii): $\xi(t,t-k) \stackrel{\text{def}}{=} \xi(t,k)$, where $\mathbf{\Phi}_{t,k}^{(1)} = \mathbf{\Phi}_{t,k}$ and $\xi^1(t,k) \stackrel{\text{def}}{=} \xi(t,k)$ we drop the superscript 1 for notational convenience), as introduced in eqs. (5) and (6), and

$$\mathbf{\Phi}_{t,k}^{(2)} = \begin{pmatrix} \varphi_2(\tau+1) & -1 & & & \\ & \varphi_1(\tau+2) & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_2(t-1) & \varphi_1(t-1) & -1 \\ & & & \varphi_2(t) & \varphi_1(t) \end{pmatrix}.$$

In the following Proposition we use mathematical induction to verify the above generalization formally.

Proposition 13 *The general terms of the fundamental solution sequences* $\xi_\tau^{(m)}$, $m =$

1,2*, are given by eq. (A.3), that is*

$$\xi^{(2)}(t,k) = det \begin{pmatrix} \varphi_2(\tau+1) & -1 & & & \\ & \varphi_1(\tau+2) & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_2(t-1) & \varphi_1(t-1) & -1 \\ & & & \varphi_2(t) & \varphi_1(t) \end{pmatrix} \quad \text{(A.4)}$$

and

$$\xi(t,k) = det \begin{pmatrix} \varphi_1(\tau+1) & -1 & & & \\ \varphi_2(\tau+2) & \varphi_1(\tau+2) & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_2(t-1) & \varphi_1(t-1) & -1 \\ & & & \varphi_2(t) & \varphi_1(t) \end{pmatrix}. \quad \text{(A.5)}$$

Proof If t $=\tau+1$ and $t=\tau+2$ then $\xi(\tau+1,1)$ and $\xi(\tau+2,2)$ is the third term and fourth term of the sequences as directly verified by eq. (A.2). We assume that $\xi(t-2,k-2)$ and $\xi(t-1,k-1)$ are terms of $\xi_\tau^{(1)}$. We show that $\xi(t,k)$ is also a term of $\xi_\tau^{(1)}$. Expanding $\xi(t,k)$ along the last row and taking into account that $\mathbf{\Phi}_{t,k}$ is a $k \times k$ matrix, we have:

$$\xi(t,k) = (-1)^{2k}\varphi_1(t)\, det \begin{pmatrix} \varphi_1(\tau+1) & -1 & & & \\ \varphi_2(\tau+2) & \varphi_1(\tau+2) & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_2(t-2) & \varphi_1(t-2) & -1 \\ & & & \varphi_2(t-1) & \varphi_1(t-1) \end{pmatrix} +$$

$$(-1)^{2k-1}(-1)\varphi_2(t)\, det \begin{pmatrix} \varphi_1(\tau+1) & -1 & & & \\ \varphi_2(\tau+2) & \varphi_1(\tau+2) & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \varphi_2(t-3) & \varphi_1(t-3) & -1 \\ & & & \varphi_2(t-2) & \varphi_1(t-2) \end{pmatrix}.$$

Using the induction hypothesis, the above result can be written as

$$\xi(t,k) = \varphi_1(t)\xi(t-1,k-1) + \varphi_2(t)\xi(t-2,k-2),$$

which shows that $\xi(t,k)$ is a homogeneous solution of eq. (2). Thus $\xi(t,k)$ in eq. (A.5) is a term of the solution sequence and the induction is complete. By analogy, we can show eq. (A.4) and the proof is complete. ■

The fundamental solution $\xi(t,k)$ (respectively $\xi^2(t,k)$) can be obtained by augmenting the core solution matrix $\boldsymbol{C}_{t,k}$ (see eq. (4) in the main body of the paper) on the left by a $k \times 1$ column consisting of the first k entries of the second column (respectively of the first column) of **P**or **Φ** .

Proof (**of Theorem 1**) As a direct consequence of Proposition 13, the general homogeneous solution of eq. (2) is the linear combination of the fundamental solutions as given below:

$$y_{t,k}^{hom} = \xi(t,k)y_\tau + \xi^{(2)}(t,k)y_{\tau-1}. \tag{A.6}$$

By expanding $\xi^2(t,k)$ along the first column we obtain

$$\xi^{(2)}(t,k) = \varphi_2(\tau+1)\xi(t,k-1)$$

and therefore eq. (A.6) takes the form

$$y_{t,k}^{hom} = \xi(t,k)y_\tau + \varphi_2(\tau+1)\xi(t,k-1)y_{\tau-1}\,,$$

which coincides with the general homogeneous solution employed in eq. (8). Next, we show that $y_{t,k}^{par}$, employed in eq. (8), is a particular solution of eq. (2). Using the same arguments as in the proof of Proposition 13 we can show that

$$y_{t,k}^{par} = det\begin{pmatrix} \varphi_0(\tau+1)+\varepsilon_{\tau+1} & -1 & & & \\ \varphi_0(\tau+2)+\varepsilon_{\tau+2} & \varphi_1(\tau+2) & -1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \varphi_0(t-1)+\varepsilon_{t-1} & & \varphi_2(t-1) & \varphi_1(t-1) & -1 \\ \varphi_0(t)+\varepsilon_t & & & \varphi_2(t) & \varphi_1(t) \end{pmatrix}, \tag{A.7}$$

is the solution of the initial value problem determined by eq. (2) subject to the initial values $y_{-1} = y_0 = 0$. This is the determinant of the core solution matrix $\boldsymbol{C}_{t,k}$ augmented on the left by a $k \times 1$ column consisting of the opposite sign first k entries of the right-hand side sequence of eq. (2).
Now expanding the determinant in eq. (A.7) along the first column we obtain $y_{t,k}^{par}$ in terms of $\xi(t,i)$ and $\varphi_0(t\text{-}i)$, ε_{t-i} for $i = 0,1,\ldots,k\text{-}1$, as used in eq. (8). Therefore, the general solution in eq. (8), as the sum of the general homogeneous solution plus a particular solution, has been established. This completes the proof of Theorem 1. ■

B APPENDIX

In this Appendix we will prove Proposition 2 by mathematical induction. For n+2 the result has been proved in eq. (14). If we assume it holds for n then it will be sufficient to prove that it holds for n+1 as well.

Proof (**Proposition** 2) Assume that

$\xi_{s,nl} = |\mathbf{\Phi}_{s,nl}| = \sum_{i_1=0}^{1} \cdots \sum_{i_{n-1}=0}^{1} \{\xi(s, l - i_1)[\prod_{g=2}^{n-1} \phi_{g-1}\xi(s - i_{g-1}, l - i_g - i_{g-1})]\phi_{n-1}\xi(s - i_{n-1}, l - i_{n-1})\}$.

Similarly to eq. (14) we can express $\xi(s,(n+1)l)$as the determinant of a 2×2 block matrix:

$$\xi(s,(n+1)l) = \begin{vmatrix} \mathbf{\Phi}_{s,l} & \mathbf{0}_{l\times nl} \\ \mathbf{0}_{s,nl,l} & \mathbf{\Phi}_{s,nl} \end{vmatrix} = |\mathbf{\Phi}_{s,nl}||\mathbf{\Phi}_{s,l}| + \varphi_{2,s+1}|\mathbf{\Phi}_{s,nl-1}||\mathbf{\Phi}_{s-1,l-1}|$$
$$= \xi(s,nl)\xi(s,l) + \varphi_{2,s+1}\xi(s,nl-1)\xi(s-1,l-1), \tag{B.2}$$

where $\mathbf{0}_{s,nl,l}$, is a $nl \times l$ matrix of zeros except for $\varphi_{2,s+1}$ in its $1\times l$ entry, $\mathbf{0}_{l\times nl}$ an $l \times nl$ matrix of zeros. Combining eqs. (B.1) and (B.2) yields

$$\xi(s,(n+1)l) = \sum_{i_1=0}^{1} \cdots \sum_{i_{n-1}=0}^{1} \{\xi(s, l - i_1)[\prod_{g=2}^{n-1} \phi_{g-1}\xi(s - i_{g-1}, l - i_g - i_{g-1})]\phi_{n-1}\xi(s - i_{n-1}, l - i_{n-1})\}\xi(s,l) + \sum_{i_1=0}^{1} \cdots \sum_{i_{n-1}=0}^{1} \{\xi_{s,l-i_1}[\prod_{g=2}^{n-1} \phi_{g-1}\xi(t - i_{g-1}, l - i_g - i_{g-1})]\phi_{n-1}\xi(s - i_{n-1}, l - 1 - i_{n-1})\}\varphi_{2,s+1}\xi(s-1,l-1)$$

which completes the proof. ■

C APPENDIX

Vector Seasons Representation

For the benefit of the reader this Appendix reviews some results on PARMA models. We recall that the drift and the autoregressive coefficients are periodically varying: $\varphi_m(t) = \varphi_m(t_s)$, $m = 1,2$. We also recall that denotes time at t_s the s-th season: t_s=$(T\text{-}1)l+s$, s = 1,...,l, and that we can write $\varphi_m(t_s) = \varphi_{ms}$ (see eq. (9)).

A convenient representation of the PAR(2;l) model (in eq. (9)) is the VAR(1) representation- hereafter we will refer to it as the vector of seasons (VS) representation (see, for example, Tiao and Guttman, 1980; Osborn, 1991; Franses, 1994, 1996a,b; Del Barrio Castro and Osborn, 2008).

The corresponding VS representation of the PAR(2;l) model (ignoring the drifts) is given by

$$\mathbf{\Phi}_0\mathbf{y}_T = \mathbf{\Phi}_1\mathbf{y}_{T-1} + \boldsymbol{\varepsilon}_T, \tag{C.1}$$

With $\mathbf{y}_T = (y_{1T}, \ \cdots, \ y_{lT})'$, $\boldsymbol{\varepsilon}_T = (\varepsilon_{1T}, \ \cdots, \ \varepsilon_{lT})'$, where the first subscript refers to the season (s) and the second one to the period (T). Moreover, $\mathbf{\Phi}_0$ is an $l \times l$ parameter matrix whose (i,j) entry is:

$$\begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad j > i, \\ -\varphi_{i-j,i} & if \quad j < i, \end{cases}$$

and $\mathbf{\Phi}_1$ is an $l \times l$ parameter matrix with (i, j) elements $\varphi_{i+l-j,i}$ (see, for example, Lund and Basawa, 2000, and Franses and Paap, 2005).

As pointed out by Franses (1994), the idea of stacking was introduced by Gladyshev (1961) and is also considered in e.g., Pagano (1978). Tiao and Guttman (1980), Osborn (1991) and Franses (1994) used it in the AR setting. The dynamic system in eq. (C.1) can be written in a compact form

$$\mathbf{\Phi}(B)\mathbf{y}_T = \boldsymbol{\varepsilon}_T \text{ or } |\mathbf{\Phi}(B)|\mathbf{y}_T = adj[\mathbf{\Phi}(B)]\boldsymbol{\varepsilon}_{\mathrm{T},}$$

where $\mathbf{\Phi}(B) = \mathbf{\Phi}_0 - \mathbf{\Phi}_1(B)$. Stationarity of y_T requires the roots of $|\boldsymbol{\Phi}(z^{-1})|$=0 to lie strictly inside the unit circle (see, among others, Tiao and Guttman, 1980, Osborn, 1991; Franses, 1994, 1996a; Franses and Paap, 2005; Del Barrio Castro and Osborn, 2008).

As an example, consider the PAR(2; 4) model for which the characteristic equation is

$$|\mathbf{\Phi}_0 - \mathbf{\Phi}_1 z| = \begin{vmatrix} 1 & 0 & -\varphi_{2,1}z & -\varphi_{1,1}z \\ -\varphi_{1,2} & 1 & 0 & -\varphi_{2,2}z \\ -\varphi_{2,3} & -\varphi_{1,3} & 1 & 0 \\ 0 & -\varphi_{2,4} & -\varphi_{1,4} & 1 \end{vmatrix} = 0.$$

Hence, when the nonlinear parameter restriction

$$\begin{gathered}\big|\varphi_{2,2}\varphi_{1,3}\varphi_{1,4} + \varphi_{2,2}\varphi_{2,4} + \varphi_{2,1}\varphi_{1,2}\varphi_{1,3} + \varphi_{2,1}\varphi_{2,3} + \varphi_{1,1}\varphi_{1,2}\varphi_{1,3}\varphi_{1,4} \\ +\varphi_{1,1}\varphi_{1,2}\varphi_{2,4} + \varphi_{1,1}\varphi_{1,4}\varphi_{2,3} - \varphi_{2,1}\varphi_{2,2}\varphi_{2,3}\varphi_{2,4}\big| < 1,\end{gathered}$$

is imposed on the parameters, the VS representation of the PAR(2;4) model is stationary (see Franses and Paap, 2005). When $\varphi_{2,s} = 0$ for all s, that is we have the PAR(1;4) model, then the stationarity condition reduces to: $|\varphi_{1,1}\varphi_{1,2}\varphi_{1,3}\varphi_{1,4}|<1$, which is equivalent to our condition $|\xi_{t,l}| < 1$, or in other words, that the absolute value of $|\mathbf{\Phi}(l)|$ is less than one.