

# Explicit and Compact Representations for the One-Sided Green's Function and the Solution of Linear Difference Equations with Variable Coefficients\*

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## Abstract

Leibniz' combinatorial formula for determinants is modified to establish a condensed and easily handled compact representation for Hessenbergians, referred to here as Leibnizian representation. Alongside, the elements of a fundamental solution set associated with linear difference equations with variable coefficients of order  $p$  are explicitly represented by  $p$  banded Hessenbergian solutions, built up solely of the variable coefficients. This yields banded Hessenbergian representations for the elements both of the product of companion matrices and of the determinant ratio formula of the one-sided Green's function (Green's function for short). Combining the above results, the elements of the foregoing notions are endowed with compact representations formulated here by Leibnizian and nested sum representations. We show that the elements of the fundamental solution set can be expressed in terms of the first banded Hessenbergian fundamental solution, called principal determinant function. We also show that the Green's function coincides with the principal determinant function, when both functions are restricted to a fairly large domain. These results yield, an explicit and compact representation of the Green's function restriction along with an explicit and compact solution representation of the previously stated type of difference equations in terms of the variable coefficients, the initial conditions and the forcing term. The equivalence of the Green's function solution representation and the well known single determinant solution representation is derived from first principles. Algorithms and automated software are employed to illustrate the main results of this paper.

**Keywords:** Green's function, Linear difference equation, Variable coefficients, Linear recurrence, Compact representation, Hessenberg matrix, Hessenbergian, Fundamental set, ARMA models.

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## 1. Introduction

Linear difference equations with variable coefficients of order  $p$  (briefly VC-LDEs( $p$ )) are broadly used to model discrete-time non-stationary stochastic processes such as autoregressive moving average (ARMA) models with time-dependent coefficients. This type of models, compared to those with constant coefficients, turn out to be more realistic and sensitive to abrupt and structural changes, as they are efficient approximations to non-linear ones, while linearity maintains interpretation and forecasting advantages (see [1]). Efficient explicit representations to the solution of VC-LDEs( $p$ ) having order greater than one ( $p > 1$ ) is a long-standing research topic. There are two dominant schemes for an explicit solution representation of VC-LDEs( $p$ ), those of determinant representation (see [2, 3]) and those of compact representation (see [4, 5]). A pioneer work to link the two representation schemes was recently accomplished by Marrero and Tomeo in [6], establishing there the equivalence between the combinatorial solution representation of VC-LDEs( $p$ ) obtained by Mallik in [4] and the single determinant solution representation obtained by Kittappa in [3]. They also established in [6] a nested sum representation for Hessenbergians and, as a consequence, a compact solution representation of VC-LDEs( $p$ ).

In this context, we provide here a more condensed combinatorial representation for Hessenbergians, referred to as *Leibnizian representation* (see eq. (24) in Theorem 1). An algorithm for the symbolic computation of the Leibnizian representation of Hessenbergians is provided in Appendix C, Algorithm 1. Unlike the Leibniz combinatorial formula for  $k$ th order determinants, which consists of  $k!$  singed elementary products (SEPs) and their summation index ranges over the symmetric group of permutations, the Leibnizian representation of

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Hessenbergians, obtained here, is a sum of  $2^{k-1}$  distinct non-trivial SEPs, whose summation index ranges over integer intervals. The latter has profound effects to the asymptotic properties of the solutions of VC-LDEs( $p$ ).

Judging from the missing cross references, the solution representations obtained in the above cited references have not been utilized in time series modelling. In contrast, the large family of time-varying coefficient models, such as ARMA processes with variable coefficients (TV-ARMA), employ the one-sided Green's function representation (Green's function for short) of a particular solution of the associated VC-LDE, because it facilitates the development of elegant and generic expressions for their fundamental properties, including the Wold-Cramér solution representation, the asymptotic properties as well as the optimal linear forecasts. This approach was originated by Miller [8] (see also [9, 10, 11]) as an alternative to the standard characteristic polynomial formulation, which has been broadly used in the solution representation of LDEs( $p$ ) and associated ARMA models with constant coefficients, but loses its strength in the presence of variable coefficients (see [9]).

Explicit representations of the Green's function strongly depend upon the availability of a fundamental set of solutions (see [8], eq. (2.6), p. 39 or [24], eq. (2.11.7), p. 77)), whose elements (called linearly independent or fundamental solutions) must be explicitly expressed and computationally tractable.<sup>1</sup> The lack, in the general case, of a fundamental solution set associated with VC-LDEs( $p$ ), whose elements fulfil the above mentioned characteristics, has led to a dichotomy between the explicit representation of the properties of TV-ARMA models in terms of the Green's function and the recursive computation of this function (see for example [9]).

The two research paths in the literature concerning solution representations of VC-LDEs( $p$ ) and corresponding representations of TV-ARMA models, respectively, have been increased over time, but the overlap between them has not. The results of the present paper establish common ground developments between them and provide the mathematical framework for a unified theory of TV-ARMA models including processes with deterministic or stochastic variable coefficients (see [13]). An application of this theory is the modelling of stock volatilities during financial crises, presented in [14].

In the present research, we introduce a fundamental set of solutions associated with VC-LDEs( $p$ ), whose terms are banded Hessenbergians initiated by  $p$  distinct unit vectors of the same magnitude  $p$ , respectively (see Subsection 3.2, Proposition 2 and Theorem 2). The nonzero entries of the associated Hessenberg matrices are the variable coefficients of a VC-LDE( $p$ ) evaluated at consecutive point instances. The banded Hessenbergian form of the aforementioned  $p$  fundamental solutions is strongly suggested by their simultaneous construction, as a result of the infinite Gaussian elimination algorithm (see in Subsection 3.1). Banded Hessenbergians are computationally tractable due to the linear time complexity needed for their evaluation (see the discussion below Corollary 2). The first fundamental solution gives rise to the *principal determinant function*, denoted by  $\xi_{t,r}$  (see Definition 2). In Proposition 4 we show that the elements of the aforementioned fundamental solution set, and therefore of the general homogeneous solution of VC-LDEs( $p$ ), can be expressed in terms of the function  $\xi_{t,r}$ . A particular solution is also expressed as a linear combination of  $\xi_{t,r}$  times the forcing terms of the VC-LDE( $p$ ) (see eq. (58) in Proposition 5) and therefore as a Hessenbergian, but not, in the general case, as a banded one.

Two of the main results of this paper concern explicit and compact representations of the Green's function. The first, recovers the defining formula of the Green's function, usually denoted by  $H(t,r)$ , as a ratio of two determinants, but now their elements are banded Hessenbergians built up solely of the variable coefficients (see Theorem 4). The second, in Theorem 5, shows that the restriction of the principal determinant function  $\xi_{t,r}$  on a suitably defined domain  $\mathcal{Z}$ , coincides with the restriction of the Green's function on  $\mathcal{Z}$ . It turns out that  $\mathcal{Z}$  exactly matches the domain restriction of  $\xi_{t,r}$ , involved in the solution of VC-LDEs. This result allows us to exchange the roles between  $\xi_{t,r}$  and  $H(t,r)$  in the solution formulas of VC-LDEs( $p$ ) (see Corollary 2). To the extend of our Knowledge there are no fully explicit representations of the homogeneous and non-homogeneous solutions of VC-LDEs( $p$ ) (see eqs. (56) and (61) respectively), exclusively in terms of the Green's function, the variable coefficients, the initial conditions and the forcing terms. As a consequence, the main notions associated with VC-LDEs are solely expressed in terms of  $\xi_{t,r}$ , including the fundamental set of solutions (see Proposition 4), the product of companion matrices (see Theorem 3), the Green's function (see eq. (54)) and the homogeneous and nonhomogeneous solutions (see eqs. (43) and (60), respectively). By replacing the Green's function with the principal determinant function in the solution formulas, employed by the previously cited works on time varying coefficient models, the above noted dichotomy is reconciled (see [13]).

In Proposition 6 we show from first principles the equivalence between the Green's function solution representation and the single determinant solution representation obtained in [3]. This equivalence result capitalizes on an alternative building process yielding the same fundamental solution set, but originated by Cramer's rule rather than the infinite Gaussian elimination algorithm (see the discussion below Proposition 6).

Thanks to the Leibnizian and nested sum representations of Hessenbergians, the paper concludes with the compact representations of the Green's function restriction, involved in the general solution of VC-LDEs( $p$ ), and of the solution itself (see Section 6). In Algorithm 2 of Appendix the Leibnizian compact representation of the Green's function and the solution of VC-LDEs( $p$ ) are verified by a symbolic computation.

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<sup>1</sup>An explicit expression evaluated in polynomial running time will be referred to as computationally tractable.

## 2. Leibnizian Representation of Hessenbergians

In all that follows the set  $\mathbb{Z}$  (resp.  $\mathbb{Z}_a$ ) stands for the set of integers (resp. the set of integers greater than or equal to  $a \in \mathbb{Z}$ ) and  $\mathbb{C}$  for the algebraic field of complex numbers. The group of permutations on  $\{1, 2, \dots, k\}$  is denoted by  $\mathbb{S}_k$  and the signature,  $sgn(\ell)$ , of  $\ell \in \mathbb{S}_k$  is assigned to  $-1$  if  $\ell$  is an odd permutation of  $\mathbb{S}_k$  and  $+1$  if  $\ell$  is an even one. The building blocks of the well known Leibnizian determinant expansion for the  $k$ th order square matrix  $\mathbf{A} = [a_{i,j}]_{1 \leq i,j \leq k}$  over  $\mathbb{C}$ , that is

$$\det(\mathbf{A}) = \sum_{\ell \in \mathbb{S}_k} sgn(\ell) \prod_{i=1}^k a_{i,\ell_i}, \quad (1)$$

are the signed elementary products, which can be formally defined as follows: Let  $\ell \in \mathbb{S}_k$ . A signed elementary product (SEP) of a square matrix  $\mathbf{A} = [a_{i,j}]_{1 \leq i,j \leq k}$  over  $\mathbb{C}$  is an ordered pair  $(\ell, sgn(\ell) \prod_{i=1}^k a_{i,\ell_i})$  in  $\mathbb{S}_k \times \mathbb{C}$ , where the second component of the ordered pair is the numerical value of the SEP in  $\mathbb{C}$ . We infer that two SEPs of  $\mathbf{A}$ , say  $(\ell, sgn(\ell) \prod_{i=1}^k a_{i,\ell_i})$  and  $(l, sgn(l) \prod_{i=1}^k a_{i,l_i})$ , are equal if and only if  $\ell = l$ . In all that follows we shall use the standard notation of SEPs:  $sgn(\ell) a_{1,\ell_1} a_{2,\ell_2} \dots a_{k,\ell_k}$ ,  $\ell \in \mathbb{S}_k$ . The set of SEPs associated with  $\mathbf{A}$  will be denoted here as  $\mathcal{S}_{\mathbf{A}}$ . As a consequence of the above discussion, every SEP in  $\mathcal{S}_{\mathbf{A}}$  is associated with a permutation  $\ell$  up to the bijection:

$$\mathbb{S}_k \ni \ell \mapsto sgn(\ell) a_{1,\ell_1} \dots a_{k,\ell_k} \in \mathcal{S}_{\mathbf{A}}. \quad (2)$$

It follows from the bijection in (2) that the number of distinct SEPs in  $\mathcal{S}_{\mathbf{A}}$  is  $k!$ , since  $\text{card}(\mathbb{S}_k) = k!$ .

The  $k$ th order lower Hessenberg matrix  $\mathbf{H}_k = [h_{i,j}]_{1 \leq i,j \leq k}$  and the infinite order lower Hessenberg matrix  $\mathbf{H} = [h_{i,j}]_{i,j \geq 1}$  over  $\mathbb{C}$  are both satisfy the defining condition  $h_{i,j} = 0$ , whenever  $j - i > 1$ , and displayed below:

$$\mathbf{H}_k = \begin{bmatrix} h_{1,1} & h_{1,2} & 0 & \dots & 0 & 0 \\ h_{2,1} & h_{2,2} & h_{2,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{k-1,1} & h_{k-1,2} & h_{k-1,3} & \dots & h_{k-1,k-1} & h_{k-1,k} \\ h_{k,1} & h_{k,2} & h_{k,3} & \dots & h_{k,k-1} & h_{k,k} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} h_{1,1} & h_{1,2} & 0 & 0 & \dots \\ h_{2,1} & h_{2,2} & h_{2,3} & 0 & \dots \\ h_{3,1} & h_{3,2} & h_{3,3} & h_{3,4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}. \quad (3)$$

From here onwards a matrix  $\mathbf{H}_k$  is considered as a term of the infinite chain of lower Hessenberg matrices  $\mathbf{H}_1 \sqsubset \mathbf{H}_2 \sqsubset \dots \sqsubset \mathbf{H}_k \sqsubset \dots \sqsubset \mathbf{H}$ , where the notation  $\mathbf{H}_k \sqsubset \mathbf{H}$  means that  $\mathbf{H}_k$  is a top submatrix of  $\mathbf{H}$  consisting of the first  $k$  rows and columns of  $\mathbf{H}$ . The determinant of  $\mathbf{H}_k$  for  $k \geq 1$  satisfies the well known recurrence

$$\det(\mathbf{H}_k) = h_{k,k} \det(\mathbf{H}_{k-1}) + \sum_{i=1}^{k-1} (-1)^{k-i} h_{k,i} \prod_{j=i}^{k-1} h_{j,j+1} \det(\mathbf{H}_{i-1}), \quad (4)$$

where  $\det(\mathbf{H}_0) = 1$  and  $\det(\mathbf{H}_1) = h_{1,1}$  (for a proof of the recurrence formula in eq. (4) see [15]).

The zero valued entries of  $\mathbf{H}_k$  positioned above the superdiagonal, that is the entries  $h_{i,j}$  whose indices satisfy  $j - i > 1$ , will be called *trivial*, while the remaining entries of  $\mathbf{H}_k$ , including the entries of the superdiagonal, will be called *non-trivial*. A SEP of  $\det(\mathbf{H}_k)$  will be called *trivial* if it contains at least one trivial entry. Otherwise it is called *non-trivial*. Throughout this paper the set of distinct non-trivial SEPs associated with  $\det(\mathbf{H}_k)$  is denoted by  $\mathcal{E}_k$ . If  $i, j \in \mathbb{Z}$ , we adopt the integer interval notation:  $[[i, j]] \stackrel{\text{def}}{=} [i, j] \cap \mathbb{Z}$  and  $\mathbb{I}_{k-1} \stackrel{\text{def}}{=} [[0, 2^{k-1} - 1]]$ .

### 2.1. Non-trivial SEPs and their String Structure

The non-trivial entries  $h_{i,j}$ ,  $j \leq i$ , positioned below and including the main diagonal of  $\mathbf{H}$ , will be called *standard factors*, while the sign-opposite entries of the super-diagonal, i.e. the entries  $-h_{i,i+1}$ , will be called *non-standard factors*. By assigning  $c_{i,j} = h_{i,j}$ , whenever  $j \neq i + 1$ , and  $c_{i,i+1} = -h_{i,i+1}$  the matrices in eqs. (3) take the form:

$$\mathbf{H}_k = \begin{bmatrix} c_{1,1} & -c_{1,2} & 0 & \dots & 0 & 0 \\ c_{2,1} & c_{2,2} & -c_{2,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{k-1,1} & c_{k-1,2} & c_{k-1,3} & \dots & c_{k-1,k-1} & -c_{k-1,k} \\ c_{k,1} & c_{k,2} & c_{k,3} & \dots & c_{k,k-1} & c_{k,k} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} c_{1,1} & -c_{1,2} & 0 & 0 & \dots \\ c_{2,1} & c_{2,2} & -c_{2,3} & 0 & \dots \\ c_{3,1} & c_{3,2} & c_{3,3} & -c_{3,4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}. \quad (5)$$

Writing the Hessenbergian recurrence in eq. (4) in terms of the entries of  $\mathbf{H}_k$  in eq. (5), after some algebraic manipulations (see for details Proposition A1 (i) in Appendix) can be equivalently rewritten as

$$\det(\mathbf{H}_k) = c_{k,k} \det(\mathbf{H}_{k-1}) + \sum_{j=1}^{k-1} \prod_{i=j}^{k-1} c_{k,i} c_{i,i+1} \det(\mathbf{H}_{j-1}). \quad (6)$$

The recurrence in eq. (6) consists exclusively of positively singed non-trivial SEPs, which gives a comparative advantage to the Hessenberg matrix forms in eqs. (5) over those in eqs. (3), respectively. The recurrence of  $\det(\mathbf{H}_k)$  in eq. (6) can be written in an expanded form as:

$$\det(\mathbf{H}_k) = c_{1,2}c_{2,3} \dots c_{k-1,k}c_{k,1} \det(\mathbf{H}_0) + c_{2,3}c_{3,4} \dots c_{k-1,k}c_{k,2} \det(\mathbf{H}_1) + c_{3,4}c_{3,5} \dots c_{k-1,k}c_{k,3} \det(\mathbf{H}_2) + \dots + c_{k-1,k}c_{k,k-1} \det(\mathbf{H}_{k-2}) + c_{k,k} \det(\mathbf{H}_{k-1}). \quad (7)$$

We show in Proposition A1 (ii) of Appendix A, that the total number of distinct non-trivial SEPs involved in the recurrence (7) and therefore in the recurrences in eqs. (4) and (6), is  $\text{card}(\mathcal{E}_k) = 2^{k-1}$ . This number is considerably less than the number  $k!$  of distinct SEPs involved in the Leibniz' determinant expansion in eq. (1).

Next, we define the notion of *strings*<sup>2</sup> associated with an infinite order Hessenberg matrix  $\mathbf{H}$  in eq. (3) or (5). Informally speaking, strings are pieces of non-trivial SEPs and pieces of strings are also strings. A formal Definition is given below:

**Definition 1** (Strings). *A finite product of consecutive factors associated with  $\mathbf{H}$  is said to be a string, denoted by  $C[j, m; \ell] = c_{j,\ell_j}c_{j+1,\ell_{j+1}} \dots c_{m,\ell_m}$ , if there is some  $k \in \mathbb{Z}_1$  and a non-trivial SEP, say  $C \in \mathcal{E}_k$ , such that  $C$  includes  $C[j, m; \ell]$ , that is:*

$$C = c_{1,\ell_1} \dots \underbrace{c_{j,\ell_j} \dots c_{m,\ell_m}}_{C[j, m; \ell]} \dots c_{k,\ell_k}.$$

An initial string determined by  $m$  and  $\ell$  is defined by  $C[1, m; \ell]$  and is shortly denoted as  $C[m; \ell]$ .

Formally the string  $C[j, m; \ell]$  is uniquely determined by  $(j, m, \ell) \in \mathbb{Z}_1^2 \times \mathbb{S}_k$  for any  $k \geq m$ . Let  $C[j, m; \ell]$  be a string. If  $j = m$ , then  $C[m, m; \ell] = c_{m,\ell_m}$ . If  $i \geq j$  and  $n \leq m$ , then Definition 1 implies that  $C[i, n; \ell]$  is also a string, since it is included in the same SEP as  $C[j, m; \ell]$ . Accordingly,  $C[i, n; \ell]$  might be said to be a *substring* of  $C[j, m; \ell]$ . Adopting the convention  $c_{0,0} = h_{0,0} = 1$ , the class of all initial strings determined by  $i \geq 0$  is denoted by  $\mathfrak{C}[i]$ . The first three classes of initial strings are:  $\mathfrak{C}[0] = \{c_{0,0}\}$ ,  $\mathfrak{C}[1] = \{c_{1,1}, c_{1,2}\}$  and  $\mathfrak{C}[2] = \{c_{1,1}c_{2,2}, c_{1,1}c_{2,3}, c_{1,2}c_{2,1}, c_{1,2}c_{2,3}\}$ . The major difference between initial strings and non-trivial SEPs is demonstrated by the following example: The initial string  $c_{1,1}c_{2,3}$  is included in the non-trivial SEP  $c_{1,1}c_{2,3}c_{3,2}$ , but the string under discussion is not a SEP. On the other hand, every non-trivial SEP is an initial string, since every SEP is included in itself. As a consequence, every string is included in an initial string.

A non-trivial element (or factor)  $c_{i,m}$  of  $\mathbf{H}$  is said to be an *immediate successor* (IS) of the initial string  $C[i-1; \ell] = c_{1,\ell_1} \dots c_{i-1,\ell_{i-1}}$ , whenever  $c_{1,\ell_1} \dots c_{i-1,\ell_{i-1}}c_{i,m}$  is an initial string too. For instance, the immediate successors of  $c_{1,1}$  are  $c_{2,2}$  and  $c_{2,3}$ . Some elementary properties of strings, as arrays of standard and non-standard factors, are summarized below:

**Proposition 1** (Properties of Strings).

- 1) Every non-trivial entry  $c_{i,j}$  of  $\mathbf{H}$  in eq. (5) is an IS of some initial string.
- 2) Every initial string of  $\mathbf{H}$  has two ISs. One of these is non-standard (therefore the other must be standard).
- 3) Let  $c_{i-1,\ell_{i-1}}$  be any standard factor of an initial string,  $C[k; \ell]$  for some  $k \geq i-1$ . Then the only possible ISs of the initial string  $C[i-1; \ell]$  are  $c_{i,i}$  and  $c_{i,i+1}$ .
- 4) Let  $c_{i,\ell_i}$  be the standard IS of the initial string  $C[i-1; \ell]$ . If the number of consecutive non-standard predecessors of  $c_{i,\ell_i}$  is  $m$ , then  $\ell_i = i - m$ .

*Proof.* 1) It follows directly from the recurrence in eq. (7). 2) Trivially, the ISs of  $c_{1,1}$  are  $c_{2,2}$  and  $c_{2,3}$  (standard and non-standard, respectively). An initial string, say  $C[i-1; \ell] = c_{1,\ell_1} \dots c_{i-1,\ell_{i-1}}$  for  $i \geq 1$  of  $\mathbf{H}$  has only two ISs, since there are  $(i+1)$  candidates, that is the number of the non-trivial elements of the  $i$ th row of  $\mathbf{H}$ , minus the number  $(i-1)$  of the preceding factors, whose column indices have already occurred in the string. Since the column index  $(i+1)$  has not previously used by preceding factors, we conclude that one of these ISs is the non-standard factor  $c_{i,i+1}$ , whence the other must be standard. 3) It suffices to show that the standard IS of  $C[i-1; \ell]$  is  $c_{i,i}$ , provided that  $c_{i-1,\ell_{i-1}}$  is standard, that is to show that  $c_{m,i}$  for any  $m = 1, 2, \dots, i-1$  is not a factor of this string. If  $i - m > 1$  (or  $i - m \geq 2$ ), then  $c_{m,i}$  is a trivial entry, because  $c_{i-2,i}, c_{i-3,i}, \dots, c_{1,i}$  are all trivial entries, and therefore are not factors of the string. Moreover, the non-standard factor  $c_{i-1,i}$  is not a factor of the string, since, by hypothesis,  $c_{i-1,\ell_{i-1}}$  is standard. Thus, the only available (not previously occurred) standard IS of  $c_{i-1,\ell_{i-1}}$  is  $c_{i,i}$ , as required. 4) Property 4 is a generalization of Property 3 and it is shown in Proposition A2 of the Appendix A (see also the tree diagram below).  $\square$

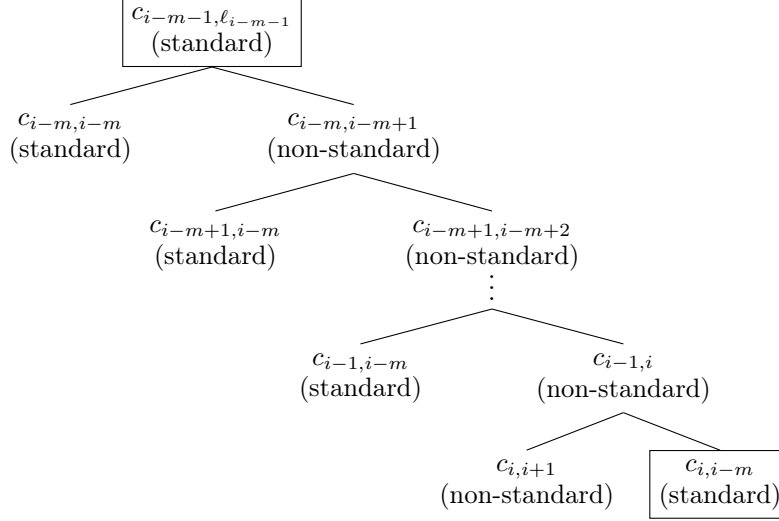
As a consequence of Property 2, the class  $\mathfrak{C}[k]$  consists of  $2^k$  initial strings, since  $\text{card}(\mathfrak{C}[k]) = 2\text{card}(\mathcal{E}_k)$ . We remark that Property 3 follows directly from Property 4, since the number of the non-standard factors between two successive standard ones in any initial string is  $m = 0$ . Property 4 can be used to identify the ISs, say  $c_{i,\ell_i}$ , of any initial string, by using the following method:

<sup>2</sup>The analysis of non-trivial SEPs of Hessenbergians via strings, defined on Hessenberg matrices of infinite order, was first introduced by the authors in [7].

**Method.** Let  $C[i-1, \ell] = c_{1, \ell_1} c_{2, \ell_2} \dots c_{i-1, \ell_{i-1}}$  be an initial string. If  $c_{i, \ell_i}$  is the non-standard IS of  $C[i-1, \ell]$ , then  $\ell_i = i+1$  and  $c_{i, \ell_i} = c_{i, i+1}$ . If  $c_{i, \ell_i}$  is the standard the IS of  $C[i-1, \ell]$ , then, in order to identify it, we need to count all consecutive non-standard predecessors of  $c_{i, \ell_i}$ . Call this number  $m$  ( $m$  could be:  $0, 1, \dots, i-1$ ). As  $c_{i-m-1, \ell_{i-m-1}}$  is a standard factor, Property 4 entails that  $\ell_i = i-m$  and the standard IS of  $C$  is:  $c_{i, \ell_i} = c_{i, i-m}$ .

**Examples:** 1) Property 3 entails that the ISs of the string  $c_{1,1}c_{2,2}$  are  $c_{3,4}$  (non-standard) and  $c_{3,3}$  (standard). In particular,  $c_{3,3}$  is the standard IS of the string, since  $c_{2,2}$  is standard. These are also the ISs of the string  $c_{1,2}c_{2,1}$ , since  $c_{2,1}$  is standard. 2) The ISs of the string  $c_{1,1}c_{2,3}$  are  $c_{3,4}$  (non-standard) and  $c_{3,2}$  (standard). To see how the latter follows from Property 4, denote the standard IS of the string  $c_{1,1}c_{2,3}$  as  $c_{3, \ell_3}$ . Since  $c_{1,1}$  is standard and  $c_{2,3}$  is non-standard,  $c_{3, \ell_3}$  has only one standard predecessor, whence  $m = 1$  and  $c_{3, \ell_3} = c_{3, 3-1} = c_{3,2}$ .

Properties 2, 3 and 4 can be visualised by a tree diagram:



## 2.2. Leibnizian Representation of Hessenbergians

The indexing function  $\sigma_{k,i}$ , yielding the Leibnizian representation of a Hessenbergian,  $|\mathbf{H}_k|$  (see eq. (24)), is defined as a composite of two functions: The outer function  $\mathfrak{A}_{k,i}$  and the inner function  $\tau_k$ . These functions are established in terms of integer functions in the next two paragraphs of the present Subsection.

### Outer Function

In what follows,  $\mathfrak{A}_k$  will stand for the set of  $k$ -arrays  $\mathbf{r} = (r_1, r_2, \dots, r_k)$  with components either  $r_i = 0$  or  $1$  for  $1 \leq i \leq k-1$  and  $r_k = 1$ . Trivially  $\mathfrak{A}_1 = (1)$  and the number of the elements in  $\mathfrak{A}_k$  is  $\text{card}(\mathfrak{A}_k) = 2^{k-1}$ . The outer building function, constructed below, considerably reduces the number of SEPs in the Leibniz determinant formula of Hessenbergians, built up solely of the non-trivial entries  $c_{i, \mathfrak{A}_{k,i}(\mathbf{r})}$  of  $\mathbf{H}_k$  for  $\mathbf{r} \in \mathfrak{A}_k$  (see eq. (16)).

We introduce the function  $f_k$ , which maps every  $C = c_{1, \ell_1} c_{2, \ell_2} \dots c_{k, \ell_k} \in \mathcal{E}_k$ , to  $\mathbf{r} = (r_1, r_2, \dots, r_{k-1}, 1) \in \mathfrak{A}_k$ , according to the rule:  $r_i = 0$ , whenever  $c_{i, \ell_i}$  is non-standard and  $r_i = 1$ , whenever  $c_{i, \ell_i}$  is standard. Since the last factor of a non-trivial SEP is always standard, the last component of the array  $\mathbf{r}$  has been assigned to 1, that is  $r_k = 1$ . As shown in Proposition A3 of the Appendix, the above rule induces the bijective mapping:

$$f_k : \mathcal{E}_k \ni C \mapsto f_k(C) \in \mathfrak{A}_k$$

For example a non-trivial SEP, say  $C = c_{1,1}c_{2,3}c_{3,2}c_{4,5}c_{5,6}c_{6,4}c_{7,7}$  is mapped uniquely to the array  $f_7(C) = (1, 0, 1, 0, 0, 1, 1)$  and vice versa. In order to verify that  $f_7^{-1}(1, 0, 1, 0, 0, 1, 1) = C$ , we first assign the 0s, which are positioned at  $i = 2, 4, 5$ , to the non-standard factors having the same positions in  $C$ , that is:  $c_{2,3}, c_{4,5}, c_{5,6}$ . By virtue of Property 4 in Proposition 1, we assign the 1s to corresponding standard factors as follows: The first 1 is mapped to  $c_{1,1}$ , as  $m = 0$ . The second 1 is assigned to  $c_{3,3-1}$ , since there is only one 0 between the first and third 1s, that is  $m = 1$ . Similarly, the third 1 must be assigned to  $c_{6,6-2}$ , since  $m = 2$ . The last 1 is assigned to  $c_{7,7}$ , since  $m = 0$  and the verification is completed.

More generally speaking, arrays in  $\mathfrak{A}_k$  represent non-trivial SEPs in  $\mathcal{E}_k$ , preserving their string structure, as arrays of standard and non-standard factors. As a consequence we have:

$$\det(\mathbf{H}_k) = \sum_{\mathbf{r} \in \mathfrak{A}_k} f_k^{-1}(\mathbf{r}). \quad (8)$$

If  $f_k^{-1}(\mathbf{r}) = c_{1, \ell_1} c_{2, \ell_2} \dots c_{i, \ell_i} \dots c_{k, \ell_k} \in \mathcal{E}_k$ , we call  $f_{k,i}^{-1}(\mathbf{r}) = c_{i, \ell_i}$ , that is the  $i$ th factor of the SEP, whence

$$f_k^{-1}(\mathbf{r}) = \prod_{i=1}^k f_{k,i}^{-1}(\mathbf{r}).$$

In what follows we assign  $r_0 = 1$ . It turns out that given  $k, i \in \mathbb{Z}$  such that  $1 \leq i \leq k$ , then for any  $\mathbf{r} \in \mathfrak{R}_k$ , we have:

$$f_{k,i}^{-1}(\mathbf{r}) = \begin{cases} c_{i,i+1}, & \text{if } r_i = 0 \text{ and } i \neq k \\ c_{i,i-m}, & \text{if } r_{i-m-1} = r_i = 1 \text{ and } r_j = 0, \text{ whenever } j \in \mathbb{Z} : i-m \leq j \leq i-1. \end{cases}$$

The above piecewise expression of  $f_{k,i}^{-1}(\mathbf{r})$  can be written in a single form as:

$$f_{k,i}^{-1}(r_1, r_2, \dots, r_{i-m-2}, r_{i-m-1}, \underbrace{0, 0, \dots, 0}_m, r_i, \dots, r_{k-1}, 1) = c_{i,i-m}, \quad m = -1, 0, \dots, i-1. \quad (9)$$

As  $m$  is the number of successive 0s between  $r_{i-m-1} = 1$  and  $r_i = 1$ , the formula in eq. (9) gives:  $f_{k,i}^{-1}(\mathbf{r}) = c_{i,i-m}$ . If  $r_i = 0$ , then  $m$  is assigned to  $m = -1$ , that is the formula in eq. (9) ignores all the predecessors of  $r_i$  and gives:  $f_{k,i}^{-1}(\mathbf{r}) = c_{i,i-(-1)} = c_{i,i+1}$ . Next, we consider two special cases: *i*) Let  $r_{i-1} = r_i = 1$ . Then  $m = 0$  and  $f_{k,i}^{-1}(\mathbf{r}) = c_{i,i}$ , which is in accord with the fact that  $\{j \in \mathbb{Z} : i \leq j \leq i-1\} = \emptyset$  and  $\text{card}(\emptyset) = 0$ . *ii*) Let  $r_i = 1$  and  $m = i-1$ . Then, on account of  $r_{i-m-1} = r_{i-(i-1)-1} = r_0 = 1$ , eq. (9) yields:

$$f_{k,i}^{-1}(\underbrace{0, 0, \dots, 0}_{i-1}, 1, r_{i+1}, \dots, r_{k-1}, 1) = c_{i,i-(i-1)} = c_{i,1}.$$

In view of eq. (2.2), the expression for Hessenbergians in eq. (8) can be rewritten as:

$$\det(\mathbf{H}_k) = \sum_{\mathbf{r} \in \mathfrak{R}_k} \prod_{i=1}^k f_{k,i}^{-1}(\mathbf{r}), \quad (10)$$

which consists of  $\text{card}(\mathfrak{R}_k) = \text{card}(\mathcal{E}_k) = 2^{k-1}$  distinct non-trivial SEPs. An equivalent expression to eq. (10), but directly in terms of the entries  $c_{i,j}$  of  $\mathbf{H}_k$  in eq. (5), is obtained in eq. (16) below. For this purpose, we introduce the function

$$\zeta_{k,i}(\mathbf{r}) \stackrel{\text{def}}{=} \begin{cases} -1 & \text{if } r_i = 0 \text{ and } i < k \\ m & \text{if } r_{i-m-1} = r_i = 1 \text{ and } r_j = 0 \text{ for all } j \in \llbracket i-m, i-1 \rrbracket, \end{cases} \quad (11)$$

which returns the number of consecutive 0s preceding the component  $r_i$  ( $i \leq k$ ) in an array  $\mathbf{r} \in \mathfrak{R}_k$ . In Proposition A4 of the Appendix, we show a single formula, equivalent to the piecewise expression of  $\zeta_{k,i}$  in eq. (11), which is additionally expressed in terms of elementary functions, as displayed below

$$\zeta_{k,i}(\mathbf{r}) = r_i(i - \max_{0 \leq j < i} \{j \cdot r_j\}) - 1, \quad i \leq k, \quad (12)$$

noting that:  $\max\{a_1, a_2\} = \frac{a_1 + a_2 + \sqrt{(a_1 - a_2)^2}}{2}$  and  $\max_{1 \leq j < i} \{a_j\} = \max\{\max\{a_1, a_2\}, a_3, \dots, a_{i-1}\}$ .

Letting  $i = 1$  in eq. (12), it follows that  $j = 0$ . Recalling the convention  $r_0 = 1$ , it follows from  $\{0 \cdot r_0\} = \{0\}$  that  $\max\{0\} = 0$ , whence

$$\zeta_{k,1}(\mathbf{r}) = r_1(1 - \max\{0\}) - 1 = r_1(1 - 0) - 1 = \begin{cases} -1 & \text{if } r_1 = 0 \\ 0 & \text{if } r_1 = 1 \end{cases} \quad \text{for all } \mathbf{r} \in \mathfrak{R}_k, \quad (13)$$

which is in accord with the Definition in eq. (11). Next, we define the function:

$$\mathfrak{z}_{k,i}(\mathbf{r}) \stackrel{\text{def}}{=} i - \zeta_{k,i}(\mathbf{r}). \quad (14)$$

It follows from eq. (9) that  $f_{k,i}^{-1}(\mathbf{r}) = c_{i,i-\zeta_{k,i}(\mathbf{r})}$ , whence:

$$f_{k,i}^{-1}(\mathbf{r}) = c_{i,\mathfrak{z}_{k,i}(\mathbf{r})}. \quad (15)$$

As a demonstrative example, let  $\mathbf{r} = (1, 0, \dots, 0, 1, 1, 0, 0, 1) \in \mathfrak{R}_k$ . Applying the Definition in eq. (11), along with eqs. (14) and (15), we get:  $\zeta_{k,1}(\mathbf{r}) = 0$ ,  $\mathfrak{z}_{k,1}(\mathbf{r}) = 1 - 0 = 1$  and  $f_{k,1}^{-1}(\mathbf{r}) = c_{1,\mathfrak{z}_{k,1}(\mathbf{r})} = c_{1,1}$ . As  $r_2 = r_3 = \dots = r_{k-5} = r_{k-2} = r_{k-1} = 0$ , we have:  $\zeta_{k,2}(\mathbf{r}) = \zeta_{k,3}(\mathbf{r}) = \dots = \zeta_{k,k-5}(\mathbf{r}) = \zeta_{k,k-2}(\mathbf{r}) = \zeta_{k,k-1}(\mathbf{r}) = -1$ . Thus,  $\mathfrak{z}_{k,2}(\mathbf{r}) = 2 - (-1) = 3, \dots, \mathfrak{z}_{k,k-5}(\mathbf{r}) = k - 5 - (-1) = k - 4$ ,  $\mathfrak{z}_{k,k-2}(\mathbf{r}) = k - 2 - (-1) = k - 1$ ,  $\mathfrak{z}_{k,k-1}(\mathbf{r}) = k - 1 - (-1) = k$  and  $f_{k,2}^{-1}(\mathbf{r}) = c_{2,\mathfrak{z}_{k,2}(\mathbf{r})} = c_{2,3}, \dots, f_{k,k-5}^{-1}(\mathbf{r}) = c_{k-5,\mathfrak{z}_{k,k-5}(\mathbf{r})} = c_{k-5,k-4}$ ,  $f_{k,k-2}^{-1}(\mathbf{r}) = c_{k-2,\mathfrak{z}_{k,k-2}(\mathbf{r})} = c_{k-2,k-1}$ ,  $f_{k,k-1}^{-1}(\mathbf{r}) = c_{k-1,\mathfrak{z}_{k,k-1}(\mathbf{r})} = c_{k-1,k}$ . As  $r_{k-3} = r_{k-4} = 1$ , we conclude that the number of preceding

consecutive 0s of  $r_{k-3}$  is zero, whence:  $\zeta_{k,k-3}(\mathbf{r}) = 0$ ,  $\mathfrak{z}_{k,k-3}(\mathbf{r}) = k-3-0 = k-3$ , and  $f_{k,k-3}^{-1}(\mathbf{r}) = c_{k-3,\mathfrak{z}_{k,k-3}}(\mathbf{r}) = c_{k-3,k-3}$ . Moreover, as the number of preceding consecutive 0s of  $r_{k-4}$  is  $k-6$ , we conclude that:  $\zeta_{k,k-4}(\mathbf{r}) = k-6$ ,  $\mathfrak{z}_{k,k-4}(\mathbf{r}) = k-4-(k-6) = 2$ , and  $f_{k,k-4}^{-1}(\mathbf{r}) = c_{k-4,\mathfrak{z}_{k,k-4}}(\mathbf{r}) = c_{k-4,2}$ . Finally, as the number of preceding consecutive 0s of  $r_k = 1$  is 2 we have:  $\zeta_{k,k}(\mathbf{r}) = 2$ ,  $\mathfrak{z}_{k,k}(\mathbf{r}) = k-2$  and  $f_{k,k}^{-1}(\mathbf{r}) = c_{k,\mathfrak{z}_{k,k}}(\mathbf{r}) = c_{k,k-2}$ . Hence,

$$f_k^{-1}(1, 0, \dots, 0, 1, 1, 0, 0, 1) = c_{1,1}c_{2,3}\dots c_{k-5,k-4}c_{k-4,2}c_{k-3,k-3}c_{k-2,k-1}c_{k-1,k}c_{k,k-2}.$$

Taking into account eq. (15) the formula in eq. (10) can be expressed as

$$\det(\mathbf{H}_k) = \sum_{\mathbf{r} \in \mathfrak{R}_k} \prod_{i=1}^k c_{i,\mathfrak{z}_{k,i}}(\mathbf{r}), \quad (16)$$

that is an expression of  $\det(\mathbf{H}_k)$  solely in terms of non-trivial entries of  $\mathbf{H}_k$  in (5).

### Inner Function

The inner function  $\tau_k$  is employed to convert integers from  $\mathbb{I}_{k-1}$  to arrays in  $\mathfrak{R}_k$ . This makes it possible to replace the indexing set  $\mathfrak{R}_k$  in eq. (16) with the integer indexing set  $\mathbb{I}_{k-1} = \llbracket 0, 2^{k-1} - 1 \rrbracket$ .

In what follows we shall make use of the conventional notation

$$\mathbf{1}_k = \underbrace{11\dots 1}_k \quad (k \text{ number of 1s}), \quad \mathbf{0}_k = \underbrace{00\dots 0}_k \quad (k \text{ number of 0s})$$

that is,  $\mathbf{1}_k$  (resp.  $\mathbf{0}_k$ ) represents a binary integer consisting of  $k$  consecutive 1s (resp. 0s). Let  $\mathcal{B}_k$  be the set of binary integers from 0 up to and including  $\mathbf{1}_k$ , that is  $\mathcal{B}_k = \{0, 1, 10, \dots, \mathbf{1}_k\}$ . The binary representation of the decimal integer  $2^k$  is the binary integer  $10^k$ , and we write:  $[2^k]_2 = 10^k \in \mathcal{B}_k$ . Thus,  $[2^k - 1]_2 = \mathbf{1}_k$  and  $\mathcal{B}_k$  consists of  $2^k$  binary integers.

In the rest of this paper  $\mathcal{Z}^k$  will stand for the set of functions from  $\{1, 2, \dots, k\}$  to  $\{0, 1\}$ , that is the set of all  $k$ -arrays of 0s and 1s:  $\mathcal{Z}^k = \{(r_1, r_2, \dots, r_{k-1}, r_k) : r_i = 0 \text{ or } 1\}$ . The set  $\mathcal{Z}^k$  can be naturally identified with the segment of the non-negative binary integers up to and including the integer  $\mathbf{1}_k$ , that is the sets  $\mathcal{Z}^k$  and  $\mathcal{B}_k$  are identified up to a bijection as follows: Let  $b \in \mathcal{B}_k$ . If  $b = 0$ , then  $\mathcal{B}_k \ni 0 \equiv \mathbf{0}_k \in \mathcal{Z}^k$ . If  $b \neq 0$ , we can write  $b = 1r_{i+1}\dots r_n\dots r_k$ , where  $r_n$  is 0 or 1 for  $i+1 \leq n \leq k$ . By adding  $(i-1)$  zero digits to the left side of  $b$ , the latter is represented as

$$\begin{array}{ccccccc} 1r_{i+1}\dots r_k & \equiv & 0 & 0 & \dots & 0 & 1r_{i+1}\dots r_k \\ & & \uparrow & & & \uparrow & \uparrow \\ \text{Binary Figures :} & & 2^{k-1} & & & 2^{k-i} & 2^0 \text{ (units)} \end{array}$$

and  $b \in \mathcal{B}_k$  turns into an element of  $\mathcal{Z}^k$  by identifying:  $\mathcal{B}_k \ni b \equiv (0, 0, \dots, 1, r_{i+1}, \dots, r_k) \in \mathcal{Z}^k$ . For example, let  $1011 \in \mathcal{B}_5$ . Adding one zero to the left side of 1011, the binary 1011 is identified with the array  $(0, 1, 0, 1, 1) \in \mathcal{Z}^5$ .

The set  $\mathfrak{R}_k$  can be defined as the subset of  $\mathcal{Z}^k$  consisting of the elements of  $\mathcal{Z}^k$  whose last component is  $r_k = 1$ , that is:  $\mathfrak{R}_k = \{\mathbf{r} \in \mathcal{Z}^k : r_k = 1\}$ . Evidently  $\text{card}(\mathcal{Z}^k) = 2^k$  and  $\text{card}(\mathfrak{R}_k) = 2^{k-1}$ . Taking into account that  $\text{card}(\mathcal{B}_{k-1}) = \text{card}(\mathfrak{R}_k) = 2^{k-1}$ , we define the bijection  $\rho_k : \mathcal{B}_{k-1} \mapsto \mathfrak{R}_k$ :

$$\rho_k(\underbrace{00\dots 01r_{i+1}\dots r_{k-1}}_{k-1}) = (\underbrace{0, 0, \dots, 0, 1, r_{i+1}, \dots, r_{k-1}, 1}_k) \quad \text{and} \quad \rho_k(\underbrace{00\dots 0}_{k-1}) = (\underbrace{0, 0, \dots, 0, 1}_k) \quad (17)$$

By identifying  $\mathcal{B}_{k-1}$  with  $\mathfrak{R}_k$  via  $\rho_k$ , the function  $f_k^{-1}$ , defined by (2.2), associates every binary integer  $\mathbf{r}$  in  $\mathcal{B}_{k-1}$  with the SEP  $f_k^{-1}(\mathbf{r})$  in one-to-one fashion.

Let  $x \in \mathbb{Z}_0$  and  $y \in \mathbb{Z}_1$ . The largest integer less than or equal to the rational number  $x/y$  is usually denoted as  $\lfloor x/y \rfloor$ , where " $\lfloor \cdot \rfloor$ " is the floor function. Therefore,  $\lfloor x/y \rfloor$  coincides the quotient of the Euclidean division of  $x$  by  $y$ . Moreover,  $x \bmod y$  stands for the remainder of the division of  $x$  by  $y$ . Thus  $n \bmod 2 = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$ . If  $x - y = nk$  for some  $n \in \mathbb{Z}$ , then  $x, y$  is said to be congruent modulo  $k$ , that is  $x, y$  have the same remainder when divided by  $k$  and we write for it:  $x \equiv y \pmod{k}$ .

The conventional method for converting decimal integers into binary is based on the Euclidean division, which is applied here for converting a decimal integer  $n \in \mathbb{I}_{k-1}$  for  $k \geq 2$  into a binary  $[n]_2 = r_1r_2\dots r_{k-1} \in \mathcal{B}_{k-1}$ . The digits  $r_i$  (0 or 1) in  $[n]_2$  turns out to be the remainders of the following nested divisions:

$$n = 2q_{k-1} + r_{k-1}, q_{k-1} = 2q_{k-2} + r_{k-2}, \dots, q_2 = 2q_1 + r_1.$$

Taking into account that

$$q_{k-1} = \lfloor n : 2 \rfloor, q_{k-2} = \lfloor \lfloor n : 2 \rfloor : 2 \rfloor, \dots, q_1 = \underbrace{\lfloor \dots \lfloor \lfloor n : 2 \rfloor : 2 \rfloor \dots \rfloor : 2 \rfloor}_{k-1},$$





that is a composite of elementary functions defined over intervals of integers. In view of eq. (21), we can rewrite eq. (15) as:

$$f_{k,i}^{-1}(\tau_k(n)) = c_{i,\mathfrak{z}_{k,i}(\tau_k(n))} = c_{i,\sigma_{k,i}(n)} \quad \text{for } n \in \mathbb{I}_{k-1}. \quad (22)$$

Substituting  $\tau_k(n)$  for  $\mathbf{r}$  in eq. (2.2), the latter takes the form

$$f_k^{-1}(\tau_k(n)) = f_{k,1}^{-1}(\tau_k(n))f_{k,2}^{-1}(\tau_k(n)) \dots f_{k,k}^{-1}(\tau_k(n)) \in \mathcal{E}_k \quad \text{for } n \in \mathbb{I}_{k-1},$$

that also defines a bijection, since  $f_k^{-1}$ ,  $\tau_k$  are bijections. Applying eq. (22) to the foregoing function formula, the latter can be equivalently written as

$$(f_k^{-1} \circ \tau_k)(n) = c_{1,\sigma_{k,1}(n)}c_{2,\sigma_{k,2}(n)} \dots c_{k,\sigma_{k,k}(n)} \in \mathcal{E}_k \quad \text{for } n \in \mathbb{I}_{k-1}, \quad (23)$$

or in a mapping form as:

$$f_k^{-1} \circ \tau_k : \mathbb{I}_{k-1} \ni n \mapsto c_{1,\sigma_{k,1}(n)}c_{2,\sigma_{k,2}(n)} \dots c_{k,\sigma_{k,k}(n)} \in \mathcal{E}_k.$$

The latter has to be compared with the bijective mapping in eq. (2).

**Theorem 1.** *The Leibnizian representation of the  $k$ th order Hessenbergian  $\det(\mathbf{H}_k)$  in terms of non-trivial entries of  $\mathbf{H}_k$ , defined in eq. (5), is:*

$$\det(\mathbf{H}_k) = \sum_{n=0}^{2^{k-1}-1} \prod_{i=1}^k c_{i,\sigma_{k,i}(n)}. \quad (24)$$

*Proof.* Taking into account that the function in eq. (23) is bijective and starting with eq. (20), the result follows from

$$\begin{aligned} \det(\mathbf{H}_k) &= \sum_{n=0}^{2^{k-1}-1} f_k^{-1}(\tau_k(n)) \\ (\text{by eq. (23)}) &= \sum_{n=0}^{2^{k-1}-1} \prod_{i=1}^k c_{i,\sigma_{k,i}(n)}, \end{aligned}$$

as claimed.  $\square$

The compact representation of Hessenbergians in eq. (24) must be compared with the corresponding nested sum representation of Hessenbergians (see [6] or eq. (66) herein) and Mallik's combinatorial formula in [4], as adjusted for Hessenbergians in [6] (see eq. (9) therein). The formula in eq. (24) is also an explicit and compact alternative representation to the recurrence formula (4) for the  $k$ th order Hessenbergian. An algorithm for its evaluation associated with a computer program are presented in Appendix C, Algorithm 1. The program is formulated by the Mathematica symbolic language and verifies the formula (24), by yielding an identical result to the one obtained directly by algorithms evaluating determinants, including the recurrence in eq. (4).

### 3. Linear Difference Equations with Variable Coefficients

Nonhomogeneous linear difference equations with variable coefficients of order  $p \geq 1$  in normal form (in short VC-LDEs( $p$ )) are defined by recurrences of the form

$$y_t = \sum_{m=1}^p \phi_m(t)y_{t-m} + v_t, \quad (25)$$

where  $\phi_m(t)$ ,  $1 \leq m \leq p$  (variable coefficients) and  $v_t$  (forcing term) are complex valued functions defined for all  $t \in \mathbb{Z}_{s+1}$  for some fixed  $s \in \mathbb{Z}$ . We further assume that  $\phi_p(t) \neq 0$  for all  $t \in \mathbb{Z}_{s+1}$ , ensuring that eq. (25) is of order  $p$ .

By virtue of the existence and uniqueness of the solutions for an initial value problem (see [12], Theorem 2.7., p. 66), given a sequence of initial values  $\{y_{r+1-p}, \dots, y_r\}$  for  $r \geq s$  fixed, a solution of eq. (25) is defined as an explicit representation of  $y_t$  for  $t \geq r+1$ , written in terms of the initial values, the variable coefficients and the forcing term. This is due to the fact that all the parameters in eq. (25), that is the variable coefficients and the forcing term, are well defined functions for any integer  $t \geq r+1$ , whenever  $r \geq s$ . As a consequence, the solution sequence is well defined on the entire set  $\mathbb{Z}_{r+1-p}$ , whereas  $y_{r+1} = \phi_1(r+1)y_r + \phi_2(r+1)y_{r-1} + \dots + \phi_p(r+1)y_{r+1-p}$ . Hereafter, in absence of ambiguity, we adopt the conventional solution notation  $y_t$  in place of the formal notation  $y_{t,r}$  of the corresponding initial value problem, since  $r$  for  $r \geq s$  is assumed to be fixed.

### 3.1. A Unified Construction Process of a Fundamental Solution Set

In this Subsection we introduce a method for constructing simultaneously the elements (sequences) of a fundamental solution set associated with VC-LDEs( $p$ ) (see the first  $p$  column sequences in eq. (27) below). This is based upon the row-finite system representation of VC-LDEs( $p$ ) (see eq. (26) below) and the Gaussian elimination algorithm applied to it, but implemented with a rightmost pivoting. Unlike the recursive construction of the individual solution sequences of eq. (25), which necessarily takes on a sequence of  $p$  initial values, the unified process, presented here, constructs simultaneously all the fundamental solution sequences starting from their  $(p + 1)$  term, without assuming any initial values. This is due to the row canonical form of the reduced system coefficient matrix constructed by the infinite Gaussian elimination algorithm, as discussed below.

Row-finite  $\omega \times \omega$  (infinite) linear systems, in their general form, were first studied by Toeplitz [18] (1909), who extended some fundamental results, established on finite linear systems, to cover row-finite ones. Such systems also represent linear difference equations of irregular order, covering the case when  $\phi_p(t) = 0$  for some  $t \geq s + 1$ . Their solution representation was further developed by Fulkerson [19] (1951). He devised and proved the existence of a reduced form, identified here as Fulkerson's row reduced echelon form (FRREF) for any arbitrary row-finite matrix<sup>3</sup>. An FRREF of a row-finite matrix, say  $\mathbf{A}$ , satisfies three out of four postulates of finite matrices in row reduced echelon form (RREF). It turns out that  $\mathbf{A}$  and an FRREF of  $\mathbf{A}$  are left associates. Left association generalizes the notion of row-equivalence of finite matrices (see [20]). The RREF of a finite matrix is uniquely associated with the matrix, and therefore it is called row canonical form of the matrix. An FRREF of a row-finite matrix,  $\mathbf{A}$ , is a quasi-canonical form of  $\mathbf{A}$ , in the sense that two FRREFs of a row-finite matrix differ only by a permutation of rows. As a consequence, the advantages gained by the row-canonical RREFs for the solution representation of finite systems, are extended to row-finite ones by their quasi-canonical FRREF. The arguments in [19] establishing the existence of an FRREF for a row-finite matrix, invoked the countable axiom of choice. In contrast to the non-constructive nature of this axiom, a modified version of the Gauss-Jordan elimination algorithm has been recently introduced by Paraskevopoulos in [21], which constructs the FRREF of an arbitrary row-finite matrix and called infinite Gauss-Jordan elimination (IGJE) algorithm. In a companion paper (see [20]), he further developed the IGJE algorithm capitalizing on the type and the form of the general solution of row-finite linear systems. If the dimension of the column-null space of the coefficient matrix is infinite, then the IGJE algorithm yields a Schauder basis of the column-null space, relative to the Fréchet metric, otherwise it yields a finite basis of vector spaces. The latter type of basis coincides with a fundamental solution set associated with the row-finite system representation of VC-LDEs( $p$ ).

The solution sequence constructed by the IGJE algorithm is solely the result of a rightmost pivot elimination strategy. As a counter example, employing a VC-LDE(1), it is shown in the above cited reference that the conventional Gauss-Jordan elimination algorithm, implemented by a leftmost pivoting, fails to construct both a row-equivalent reduced matrix and the solution of the original VC-LDE(1). This was the main barrier preventing researchers from choosing the IGJE algorithm for solving VC-LDEs and more generally row-finite systems.

Eq. (25) can be equivalently represented by a row-finite linear system of the form:

$$\begin{bmatrix} \phi_p(r+1) & \phi_{p-1}(r+1) & \phi_{p-2}(r+1) & \dots & \phi_1(r+1) & -1 & 0 & 0 & \dots \\ 0 & \phi_p(r+2) & \phi_{p-1}(r+2) & \dots & \phi_2(r+2) & \phi_1(r+2) & -1 & 0 & \dots \\ 0 & 0 & \phi_p(r+3) & \dots & \phi_3(r+3) & \phi_2(r+3) & \phi_1(r+3) & -1 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} y_{r+1-p} \\ y_{r+2-p} \\ y_{r+3-p} \\ \vdots \\ y_r \\ y_{r+1} \\ y_{r+2} \end{bmatrix} = - \begin{bmatrix} v_{r+1} \\ v_{r+2} \\ \vdots \end{bmatrix}. \quad (26)$$

The coefficient matrix of eq. (26), say  $\mathbf{A}$ , is row-finite with elements the variable coefficients of eq. (25) evaluated at corresponding instances, starting at fixed instance  $r + 1$ . Since  $\mathbf{A}$  is in lower echelon form, the infinite Gaussian elimination part (IGE)<sup>4</sup> of the IGJE algorithm is only needed for reducing  $\mathbf{A}$  to its FRREF and the reduced matrix is unique, denoted by  $\mathbf{FRREF}(\mathbf{A})$ . Implementing the IGE with rightmost pivoting, the pivot elements are the  $(-1)$ s occupying the  $p$ th superdiagonal of  $\mathbf{A}$ . A step-by step construction of  $\mathbf{FRREF}(\mathbf{A})$  is presented in Appendix B. As shown there, the elimination process leads to a recurrence analogous to that in eq. (25).

The IGE algorithm constructs the  $\mathbf{FRREF}(\mathbf{A})$ , which is given by

$$\mathbf{FRREF}(\mathbf{A}) = \begin{bmatrix} -\xi_{r+1,r}^{(p)} & -\xi_{r+1,r}^{(p-1)} & \dots & -\xi_{r+1,r}^{(1)} & 1 & 0 & 0 & \dots \\ -\xi_{r+2,r}^{(p)} & -\xi_{r+2,r}^{(p-1)} & \dots & -\xi_{r+2,r}^{(1)} & 0 & 1 & 0 & \dots \\ -\xi_{r+3,r}^{(p)} & -\xi_{r+3,r}^{(p-1)} & \dots & -\xi_{r+3,r}^{(1)} & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \quad (27)$$

<sup>3</sup>A row-finite matrix is an  $\omega \times \omega$  infinite matrix, each row of which comprises a finite number of non-zero entries.

<sup>4</sup>A recursive alternative to the infinite Gaussian elimination algorithm is presented in [22].

An explicit form of each sequence  $\xi_{t,r}^{(m)}$  for  $1 \leq m \leq p$  and  $t \geq r+1$  in eq. (27) is established in Subsection 3.2 (see eqs. (30) and (31)) and conveniently visualized by VC-LDEs of order  $p = 2$  in Example 1 below.

For each  $m \in \llbracket 1, p \rrbracket$ , we define the  $m$ th unit vector by:  $\mathbf{e}_{p+1-m} = [\delta_{p+1-m,j}]_{1 \leq j \leq p}$ , where  $\delta_{i,j}$  is the Kronecker delta. We further define the augmented sequence  $\xi_{\cdot,r}^{(m)} \stackrel{\text{def}}{=} [\mathbf{e}_{p+1-m} \# [\xi_{r+1,r}^{(m)}, \xi_{r+2,r}^{(m)}, \dots]]'$ , where  $[\dots]'$  stands for transposition and “ $\#$ ” for concatenation of vectors. As  $\mathbf{FRREF}(\mathbf{A})$  is a row-canonical form of  $\mathbf{A}$  the results in [20] entail that: First, for any  $m \in \llbracket 1, p \rrbracket$  the sequence  $\xi_{\cdot,r}^{(m)}$ , is a solution of the homogeneous system  $\mathbf{A} \mathbf{y} = \mathbf{0}$  associated with eq. (26) (see Example 1 below). Equivalently, the terms  $\xi_{r+i,r}^{(m)}$ ,  $i \geq 1$ , of each individual  $\xi_{\cdot,r}^{(m)}$  form a homogeneous solution sequence of eq. (25) (when  $v_t = 0$  for all  $t \geq s+1$ ), taking on the components of  $\mathbf{e}_{p+1-m}$  as initial values. Second, the set  $\Xi_r = \{\xi_{\cdot,r}^{(m)}\}_{1 \leq m \leq p}$  is a fundamental solution set associated with eq. (26) (or eq. (25)). Both aforementioned results are independently established in the next Subsection. We remark that the fundamental solution sequence  $\xi_{\cdot,r}^{(1)}$  is the first element of  $\Xi_r$ , but its terms  $\xi_{t,r}^{(1)}$  for  $t \geq r+1$  occupy the  $p$ th opposite signed column of  $\mathbf{FRREF}(\mathbf{A})$ , namely:  $\xi_{\cdot,r}^{(1)} = [\mathbf{e}_p \# [\xi_{r+1,r}^{(1)}, \xi_{r+2,r}^{(1)}, \dots]]' = [0, 0, \dots, 1, \xi_{r+1,r}^{(1)}, \xi_{r+2,r}^{(1)}, \dots]'$ .

**Example 1.** In this Example we apply the IGE algorithm to a row-finite system representation of VC-LDE(2). The IGE algorithm applies to the coefficient matrix,  $\mathbf{A}$ , of eq. (26) for  $p = 2$  and simultaneously constructs the terms  $\{\xi^{(1)}(t, r)\}_{t \geq r+1}$  and  $\{\xi^{(2)}(t, r)\}_{t \geq r+1}$  of the two fundamental solutions, whereas their opposite signed terms occupy the second and the first columns of  $\mathbf{FRREF}(\mathbf{A})$ , respectively. The first three terms of  $\{\xi^{(1)}(t, r)\}_{t \geq r+1}$  are:

$$\begin{aligned} \xi_{r+1,r}^{(1)} &= \phi_1(r+1), & \xi_{r+2,r}^{(1)} &= \phi_1(r+1)\phi_1(r+2) + \phi_2(r+2), \\ \xi_{r+3,r}^{(1)} &= \phi_1(r+1)[\phi_1(r+2)\phi_1(r+3) + \phi_2(r+3)] + \phi_2(r+2)\phi_1(r+3), \dots \end{aligned}$$

(see the Appendix B for a verification of  $\mathbf{A} [0, 1, \xi_{r+1,r}^{(1)}, \xi_{r+2,r}^{(1)}, \dots]' = \mathbf{0}$ ). According to [20] the above sequence augmented on its left by the unit vector  $\mathbf{e}_2 = [0, 1]$  yields the first fundamental solution. The algorithmic outcomes are expansions of the following Hessenbergians:

$$\xi_{r+1,r}^{(1)}, \xi_{r+2,r}^{(1)} = \begin{vmatrix} \phi_1(r+1) & -1 \\ \phi_2(r+2) & \phi_1(r+2) \end{vmatrix}, \xi_{r+3,r}^{(1)} = \begin{vmatrix} \phi_1(r+1) & -1 & 0 \\ \phi_2(r+2) & \phi_1(r+2) & -1 \\ 0 & \phi_2(r+3) & \phi_1(r+3) \end{vmatrix}, \dots$$

The corresponding terms of  $\{\xi^{(2)}(t, r)\}_{t \geq r+1}$  are:

$$\xi_{r+1,r}^{(2)} = \phi_2(r+1), \quad \xi_{r+2,r}^{(2)} = \phi_2(r+1)\phi_1(r+2), \quad \xi_{r+3,r}^{(2)} = \phi_2(r+1)[\phi_1(r+2)\phi_1(r+3) + \phi_2(r+3)], \dots$$

This sequence augmented on its left by the unit vector  $\mathbf{e}_1 = [1, 0]$  yields the second fundamental solution. In this case, the algorithmic outcomes are also expansions of Hessenbergians of the form:

$$\xi_{r+1,r}^{(2)}, \xi_{r+2,r}^{(2)} = \begin{vmatrix} \phi_2(r+1) & -1 \\ 0 & \phi_1(r+2) \end{vmatrix}, \xi_{r+3,r}^{(2)} = \begin{vmatrix} \phi_2(r+1) & -1 & 0 \\ 0 & \phi_1(r+2) & -1 \\ 0 & \phi_2(r+3) & \phi_1(r+3) \end{vmatrix}, \dots$$

The terms  $\xi_{r+i,r}^{(m)}$  for  $m = 1, 2$  and  $i \geq 1$ , of each individual  $\xi_{\cdot,r}^{(m)}$  can also be constructed recursively via eq. (25) for  $p = 2$ , when  $v_{r+i} = 0$  for all  $i \geq 1$ , taking on the components of  $\mathbf{e}_2, \mathbf{e}_1$  as initial values, respectively.

These results led us to propose a generalized Definition of  $\xi_{t,r}^{(m)}$  in eqs. (30) and (31) below.

Applying the same sequence of row elementary operations, used by the IGE for the row reduction of  $\mathbf{A}$  to  $\mathbf{FRREF}(\mathbf{A})$ , but now to the sequence  $\{-v_{r+i}\}_{i \geq 1}$ , a particular solution sequence is constructed (see Appendix B). The process leads to a recurrence, which equivalently constructs the particular solution. This solution sequence is also explicitly represented by a Hessenbergian, but not a banded one, as shown in Proposition 5. The general solution is a linear combination of the fundamental solutions with coefficients arbitrary initial condition values  $y_{r+1-m} = a_m$  for  $1 \leq m \leq p$  (that is the general homogeneous solution, see Proposition 3) plus the aforementioned particular solution (see eq. (60)).

### 3.2. Fundamental and General Homogeneous Solutions

A fundamental set of solutions associated with VC-LDEs( $p$ ) plays a significant role in the explicit representation of the Green's function, the companion matrix product (or the Casorati matrix) as well as the general solution of VC-LDEs( $p$ ). The existence of such solution sets is theoretically guaranteed by the fundamental Theorem of VC-LDEs( $p$ ) (see [12] p. 74). As a consequence of the superposition principle (see the previously cited reference) the homogeneous solution of VC-LDEs( $p$ ) can be expressed as a linear combination of fundamental solutions whose coefficients are expressions involving the initial condition values.

Given some  $r \geq s$ , the homogeneous linear difference equation associated with eq. (25) is of the form

$$y_t = \sum_{m=1}^p \phi_m(t) y_{t-m}, \quad t \geq r+1, \quad (28)$$



**Proposition 2.** Let  $m \in \llbracket 1, p \rrbracket$ . The sequence  $\{\xi_{t,r}^{(m)}\}_{t \in \mathbb{Z}_{r+1}}$  for any arbitrary but fixed  $r \geq s$ , defined in eq. (30), solves eq. (28), assuming the initial condition unit vector  $[\xi_{r+1-p,r}^{(m)}, \xi_{r+2-p,r}^{(m)}, \dots, \xi_{r,r}^{(m)}] = \mathbf{e}_{p+1-m}$ .

*Proof.* Taking into account that  $c_{i,i+1} = 1$ , the recurrence in eq. (7) applied for  $i = k = t - r$  (the order of the matrix) takes the form:

$$|\mathbf{H}_i| = c_{i,1}|\mathbf{H}_0| + c_{i,2}|\mathbf{H}_1| + \dots + c_{i,i-1}|\mathbf{H}_{i-2}| + c_{i,i}|\mathbf{H}_{i-1}|. \quad (34)$$

We examine the following cases:

i) Let  $1 \leq i \leq p + 1 - m$ . This inequality can be equivalently written as  $r + 1 \leq t \leq r + p + 1 - m$ , which means that  $\Phi_{t,r}^{(m)}$  in eq. (31) is a full lower Hessenberg matrix. We can equivalently write the above inequality as  $t = r + p + 1 - m - l$ , whenever  $0 \leq l \leq p - m$ . As  $i = t - r = p + 1 - m - l$ , it follows from eq. (33) that:  $c_{i,1} = c_{t-r,1} = \phi_{p+1-m-l-1+m}(r+p+1-m-l) = \phi_{p-i}(t)$  and  $c_{i,2} = c_{t-r,2} = \phi_{p+1-m-l-2+1}(r+p+1-m-l) = \phi_{p-m-l}(t)$ . Proceeding in this way, the remaining values of  $c_{i,j}$  for  $3 \leq j \leq p$  are:  $c_{t-r,3} = \phi_{t-r-2}(t) = \phi_{p-m-l}(t), \dots, c_{t-r,t-r-1} = \phi_2(t), c_{t-r,t-r} = \phi_1(t)$ . We then replace the above results to the right-hand side of eq. (34) coupled with:  $|\mathbf{H}_0| = 1 = \xi_{r+1-m,r}^{(m)}$ ,  $|\mathbf{H}_1| = \phi_m(r+1) = \xi_{r+1,r}^{(m)}, \dots, |\mathbf{H}_{t-2-r}| = \xi_{t-2,r}^{(m)}$  and  $|\mathbf{H}_{t-1-r}| = \xi_{t-1,r}^{(m)}$ . As the left-hand side of eq. (34) can be replaced with  $|\mathbf{H}_i| = |\mathbf{H}_{t-r}| = \xi_{t,r}^{(m)}$ , it takes the form

$$\xi_{t,r}^{(m)} = \underbrace{\phi_{p-l}(t)\xi_{r+1-m,r}^{(m)} + \phi_{p-l-1}(t)\xi_{r+2-m,r}^{(m)} + \dots + \phi_{p-l+1-m}(t)\xi_{r,r}^{(m)}}_{\text{initial values}} + \phi_{p-l-m}(t)\xi_{r+1,r}^{(m)} + \dots + \phi_2(t)\xi_{t-2,r}^{(m)} + \phi_1(t)\xi_{t-1,r}^{(m)}$$

where the values of  $\xi_{r+1-m,r}^{(m)}$  up to and including  $\xi_{r,r}^{(m)}$  are initial values defined in eq. (32), that is  $\xi_{r+1-m,r}^{(m)} = 1$  and the remaining initial values are zero, whenever  $m \neq 1$ , while if  $m = 1$ , then  $\xi_{r,r}^{(1)} = \xi_{r+1-m,r}^{(1)} = 1$ .

ii) Let  $i > p + 1 - m$ . As  $t - r - 1 + m > p$ , in view of eq. (31), the matrix  $\Phi_{t,r}^{(m)}$  is a lower banded Hessenberg matrix. This means that the first values of  $c_{i,j}$  in eq. (34) starting from  $c_{i,1}$  up to and including  $c_{i,i-p}$  are zero. Applying eq. (33), we have:  $c_{i,i} = \phi_1(t)$ ,  $c_{i,i-1} = \phi_2(t), \dots, c_{i,i-p+1} = \phi_p(t)$ , which is in accord with eq. (33), since:  $c_{t-r,1} = \dots = c_{t-r,t-r-p} = 0$ . Substituting the above values of  $c_{i,j}$  and replacing  $\det(\mathbf{H}_{t-r-j})$  with  $\xi_{t-j,r}^{(m)}$  for  $0 \leq j \leq p$  in the recurrence (7), the latter takes the form:

$$\xi_{t,r}^{(m)} = \phi_p(t)\xi_{t-p,r}^{(m)} + \phi_{p-1}(t)\xi_{t-p+1,r}^{(m)} + \dots + \phi_2(t)\xi_{t-2,r}^{(m)} + \phi_1(t)\xi_{t-1,r}^{(m)}.$$

In both cases  $\xi_{t,r}^{(m)}$  satisfies eq. (28), as required.  $\square$

We conclude from the uniqueness of the initial value problem that the fundamental solutions constructed by the IGE algorithm or by recursion must all coincide with the banded Hessenbergian ones.

**Theorem 2.** For each fixed  $r \geq s$ , the set  $\Xi_r = \{\xi_{t,r}^{(1)}, \xi_{t,r}^{(2)}, \dots, \xi_{t,r}^{(p)}\}$ , consisting of  $p$  sequences defined over the same domain  $\mathbb{Z}_{r+1-p}$ , is a fundamental set of solutions associated with eq. (28).

*Proof.* Notice first that each sequence  $\xi_{t,r}^{(m)}$  (the  $m$ th element of  $\Xi_r$ ) starts with the value  $\xi_{r+1-p,r}^{(m)}$ , whence the domain of the function  $\xi_{t,r}^{(m)}$  is  $\mathbb{Z}_{r+1-p}$ . Taking into account that the elements of  $\Xi_r$  are solutions of eq. (28) (see Proposition 2), it suffices to verify that the set  $\Xi_r$  is linearly independent. Equivalently, it must be shown that the Casoratian of the matrix

$$\Xi_{t,r} = \begin{bmatrix} \xi_{t,r}^{(1)} & \xi_{t,r}^{(2)} & \dots & \xi_{t,r}^{(p)} \\ \xi_{t-1,r}^{(1)} & \xi_{t-1,r}^{(2)} & \dots & \xi_{t-1,r}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{t-p+1,r}^{(1)} & \xi_{t-p+1,r}^{(2)} & \dots & \xi_{t-p+1,r}^{(p)} \end{bmatrix} \quad (35)$$

associated with the solution set  $\Xi_r$  is nonzero for all  $t \geq r$ . Definition (30) entails that the matrix  $\Xi_{r,r}$  is the identity matrix of order  $p$ , that is  $\Xi_{r,r} = \mathbf{I}_p$ . Therefore the first Casoratian  $|\Xi_{r,r}|$  of the set  $\Xi_r$  is  $|\Xi_{r,r}| = 1 \neq 0$ . It follows that  $|\Xi_{t,r}| \neq 0$  for all  $t \geq r$  (see Lemma 1.3 in [8], applied for  $a = r - p + 1$ ) and the set  $\Xi_r$  is linearly independent. That is  $\Xi_r$  is a fundamental set of solutions of eq. (28).  $\square$

For any  $r \geq s$ , the dimension of the homogeneous solution space of  $\Phi_t(B)$  for  $t \geq r + 1$  in eq. (29) is  $p$ . As  $\Xi_r$  spans this space and  $\text{card}(\Xi_r) = p$ , it follows that the set  $\Xi_r$  is a basis of the column-null space of the coefficient matrix in eq. (26). Besides, as  $|\Xi_{t,r}| \neq 0$  for all  $t \geq r$ , Corollary 1 below, follows immediately.

**Corollary 1.** The Casorati matrix  $\Xi_{t,r}$  in eq. (35) is invertible (or non-singular) for all  $t \geq r$  and any  $r \geq s$ .

Corollary (1) is also established independently, as a consequence of Theorem 3 in Subsection 3.4.





*Proof.* Applying Lemma 1(i) to  $\xi_{t,r}^{(m)}$  in eq. (31), the cofactor expansion of  $\xi_{t,r}^{(m)}$  along the first column gives  $\xi_{t,r}^{(m)} = \phi_m(r+1)\xi_{t,r+1} + \phi_{m+1}(r+2)\xi_{t,r+2} + \dots + \phi_p(r+p+1-m)\xi_{t,r+p+1-m}$ , whence

$$\xi_{t,r}^{(m)} = \sum_{i=m}^p \phi_i(r+i+1-m)\xi_{t,r+i+1-m} = \sum_{i=1}^{p+1-m} \phi_{m-1+i}(r+i)\xi_{t,r+i},$$

as required.  $\square$

Taking into account that  $\xi_{t,r} = \sum_{i=1}^p \phi_i(r+i)\xi_{t,r+i}$ , by applying eq. (42) to eq. (36), we obtain explicit representations of the general solution  $y_t$  in eq. (28) solely in terms of the principal determinant function and any sequence of prescribed values  $\{y_{r+1-m}\}_{1 \leq m \leq p}$  for a fixed  $r \geq s$ , as follows:

$$y_t = \sum_{m=1}^p \sum_{i=1}^{p+1-m} \phi_{m-1+i}(r+i)\xi_{t,r+i}y_{r+1-m} = \xi_{t,r}y_r + \sum_{m=2}^p \sum_{i=1}^{p+1-m} \phi_{m-1+i}(r+i)\xi_{t,r+i}y_{r+1-m} \quad \text{for all } t \geq r+1. \quad (43)$$

### 3.4. Companion Matrix Product

We show in the current Subsection that the elements of the product of companion matrices associated with the difference operator in eq. (29) can be explicitly represented by banded Hessenbergians.

Let  $t \in \mathbb{Z}_{s+1}$ . The companion matrix of order  $p$  is given by

$$\mathbf{\Gamma}_t = \begin{bmatrix} \phi_1(t) & \phi_2(t) & \dots & \phi_{p-1}(t) & \phi_p(t) \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \quad (44)$$

Eq. (28) for  $r \geq s$  can be expressed, as a vector equation, by:

$$\mathbf{y}_t = \mathbf{\Gamma}_t \mathbf{y}_{t-1}, \quad t \geq r+1. \quad (45)$$

An extended Definition of the companion matrix product, including the case  $t = r$ , is given by

$$\mathbf{F}_{t,r} \stackrel{\text{def}}{=} \begin{cases} \mathbf{\Gamma}_t \mathbf{\Gamma}_{t-1} \dots \mathbf{\Gamma}_{r+1}, & \text{if } t \geq r+1 \\ \mathbf{I}_p, & \text{if } t = r. \end{cases} \quad (46)$$

$\mathbf{F}_{t,r}$  is invertible, since  $\mathbf{\Gamma}_i$  is invertible for all  $i \in \mathbb{Z}_{s+1}$ . As  $\mathbf{F}_{r,r} = \mathbf{I}_p$ , we further conclude that  $\mathbf{F}_{t,r}$  is invertible for all  $t \in \mathbb{Z}_r$  and any  $r \geq s$ . Taking into account that the matrix multiplication is non-commutative, we can alternatively use the condense notation:  $\mathbf{F}_{t,r} = \prod_{i=r}^{t-1} \mathbf{\Gamma}_{t-i+r}$ .

Let  $\mathbf{y}_r = [y_r, y_{r-1}, \dots, y_{r-p+1}]'$  be an initial condition vector associated with eq. (28). Then the unique vector solution of the corresponding initial value problem associated with eq. (28) can also be described by the vector equation:

$$\mathbf{y}_t = \mathbf{F}_{t,r} \mathbf{y}_r \quad \text{for } t \geq r. \quad (47)$$

This is an alternative interpretation to the solution in eq. (36). If  $t = r$ , then  $\mathbf{y}_r = \mathbf{I}_p \mathbf{y}_r$ , as expected.

In all that follows  $\mathbf{\Xi}_{t,r}$  stands for the Casorati matrix defined in eq. (35). Before proving the main result of this Subsection in Theorem 3, we recall an elementary result from linear algebra:

**Remark 1.** Let  $\mathbf{A}, \mathbf{B}$  be  $k \times k$  complex matrices. If  $\mathbf{A} \mathbf{x} = \mathbf{B} \mathbf{x}$  for all  $\mathbf{x} \in \mathbb{C}^k$ , then  $\mathbf{A} = \mathbf{B}$ .

**Theorem 3.** The product of companion matrices  $\mathbf{F}_{t,r}$  associated with eq. (28) coincides with the Casorati matrix  $\mathbf{\Xi}_{t,r}$  for all  $t \in \mathbb{Z}_r$  and any fixed  $r \geq s$ , given by eq. (35).

*Proof.* Let  $\mathbf{y}_r = [y_r, y_{r-1}, \dots, y_{r-p+1}]'$  be an arbitrary initial condition vector. Applying eq. (36) for  $t, t-1, \dots, t-p+1$ , the components of the solution vector  $\mathbf{y}_t$  associated with eq. (28) are given by:

$$\begin{aligned} y_t &= \xi_{t,r}^{(1)} y_r + \xi_{t,r}^{(2)} y_{r-1} + \dots + \xi_{t,r}^{(p)} y_{r-p+1} \\ y_{t-1} &= \xi_{t-1,r}^{(1)} y_r + \xi_{t-1,r}^{(2)} y_{r-1} + \dots + \xi_{t-1,r}^{(p)} y_{r-p+1} \\ &\vdots \\ y_{t-p+1} &= \xi_{t-p+1,r}^{(1)} y_r + \xi_{t-p+1,r}^{(2)} y_{r-1} + \dots + \xi_{t-p+1,r}^{(p)} y_{r-p+1}. \end{aligned}$$

The above  $p \times p$  system of linear equations can be expressed in a vector equation form as:

$$\mathbf{y}_t = \mathbf{\Xi}_{t,r} \cdot \mathbf{y}_r \quad \text{for } t \geq r. \quad (48)$$



A comparison of eqs. (47) and (48), on account of the uniqueness of the solution vector  $\mathbf{y}_t$ , implies that:  $\Xi_{t,r} \mathbf{y}_r = \mathbf{F}_{t,r} \mathbf{y}_r$  for all  $\mathbf{y}_r \in \mathbb{C}^p$ . It follows from Remark 1 that

$$\Xi_{t,r} = \mathbf{F}_{t,r}, \quad (49)$$

as asserted.  $\square$

By virtue of eq. (49) the entries of  $\mathbf{F}_{t,r}$  for  $t \geq r + 1$  are the banded Hessenbergians, whose elements are explicitly expressed in terms of the variable coefficients of eq. (28). As  $\mathbf{F}_{t,r}$  is invertible, we conclude from eq. (49) that  $\Xi_{t,r}$  is invertible too. This statement recovers the result stated in Corollary 1. In the following Example we apply Theorem 3 to the second order VC-LDE(2).

**Example 2.** In this example we consider the second order homogeneous VC-LDE:

$$y_t = \phi_1(t)y_{t-1} + \phi_2(t)y_{t-2}.$$

Let  $s \in \mathbb{Z}$  and  $\phi_2(t) \neq 0$  for all  $t \geq s + 1$ . The Definition in eqs. (30) and (31) is applied for  $r = t - 1, t - 2, t - 3$  to verify the identity in eq. (48), assuming that  $r \geq s$ .

i) If  $r = t - 1 \geq s$ , then we conclude that:  $\xi_{t,t-1} = \phi_1(t)$ ,  $\xi_{t,t-1}^{(2)} = \phi_2(t)$ ,  $\xi_{t-1,t-1} = 1$ ,  $\xi_{t-1,t-1}^{(2)} = 0$ . The associated companion matrix is given by

$$\mathbf{F}_{t,t-1} = \mathbf{\Gamma}_t = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \xi_{t,t-1} & \xi_{t,t-1}^{(2)} \\ \xi_{t-1,t-1} & \xi_{t-1,t-1}^{(2)} \end{bmatrix} = \Xi_{t,t-1}.$$

ii) If  $r = t - 2 \geq s$ , then we conclude that:

$$\xi_{t,t-2} = \begin{vmatrix} \phi_1(t-1) & -1 \\ \phi_2(t) & \phi_1(t) \end{vmatrix}, \quad \xi_{t,t-2}^{(2)} = \begin{vmatrix} \phi_2(t-1) & -1 \\ 0 & \phi_1(t) \end{vmatrix}, \quad \xi_{t-1,t-2} = \phi_1(t-1), \quad \xi_{t-1,t-2}^{(2)} = \phi_2(t-1).$$

The product of the first two companion matrices is given by

$$\begin{aligned} \mathbf{F}_{t,t-2} = \mathbf{\Gamma}_t \mathbf{\Gamma}_{t-1} &= \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1(t-1) & \phi_2(t-1) \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \phi_1(t)\phi_1(t-1) + \phi_2(t) & \phi_1(t)\phi_2(t-1) \\ \phi_1(t-1) & \phi_2(t-1) \end{bmatrix} \\ &= \begin{bmatrix} \xi_{t,t-2} & \xi_{t,t-2}^{(2)} \\ \xi_{t-1,t-2} & \xi_{t-1,t-2}^{(2)} \end{bmatrix} = \Xi_{t,t-2}. \end{aligned}$$

iii) If  $r = t - 3 \geq s$ , then we conclude that:

$$\xi_{t,t-3} = \begin{vmatrix} \phi_1(t-2) & -1 & 0 \\ \phi_2(t-1) & \phi_1(t-1) & -1 \\ 0 & \phi_2(t) & \phi_1(t) \end{vmatrix} = \phi_1(t-2)[\phi_1(t)\phi_1(t-1) + \phi_2(t)] + \phi_2(t-1)\phi_1(t),$$

$$\xi_{t,t-3}^{(2)} = \begin{vmatrix} \phi_2(t-2) & -1 & 0 \\ 0 & \phi_1(t-1) & -1 \\ 0 & \phi_2(t) & \phi_1(t) \end{vmatrix} = \phi_2(t-2)[\phi_1(t)\phi_1(t-1) + \phi_2(t)],$$

$$\xi_{t-1,t-3} = \begin{vmatrix} \phi_1(t-2) & -1 \\ \phi_2(t-1) & \phi_1(t-1) \end{vmatrix} = \phi_1(t-1)\phi_1(t-2) + \phi_2(t-1),$$

$$\xi_{t-1,t-3}^{(2)} = \begin{vmatrix} \phi_2(t-2) & -1 \\ 0 & \phi_1(t-1) \end{vmatrix} = \phi_1(t-1)\phi_2(t-2).$$

The product of the first three companion matrices is given by given by

$$\begin{aligned} \mathbf{F}_{t,t-3} = \mathbf{\Gamma}_t \mathbf{\Gamma}_{t-1} \mathbf{\Gamma}_{t-2} &= \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1(t-1) & \phi_2(t-1) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1(t-2) & \phi_2(t-2) \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \phi_1(t)\phi_1(t-1) + \phi_2(t) & \phi_1(t)\phi_2(t-1) \\ \phi_1(t-1) & \phi_2(t-1) \end{bmatrix} \begin{bmatrix} \phi_1(t-2) & \phi_2(t-2) \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \phi_1(t-2)[\phi_1(t)\phi_1(t-1) + \phi_2(t)] + \phi_1(t)\phi_2(t-1) & \phi_2(t-2)[\phi_1(t)\phi_1(t-1) + \phi_2(t)] \\ \phi_1(t-1)\phi_1(t-2) + \phi_2(t-1) & \phi_1(t-1)\phi_2(t-2) \end{bmatrix} \\ &= \begin{bmatrix} \xi_{t,t-3} & \xi_{t,t-3}^{(2)} \\ \xi_{t-1,t-3} & \xi_{t-1,t-3}^{(2)} \end{bmatrix} = \Xi_{t,t-3}. \end{aligned}$$



**Theorem 4.** The Green's function  $H(t, r)$  for  $(t, r) \in \mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$  associated the difference operator in eq. (29) can be explicitly expressed as a ratio of determinants:

$$H(t, r) = \begin{vmatrix} \xi_{t,s}^{(1)} & \xi_{t,s}^{(2)} & \cdots & \xi_{t,s}^{(p)} \\ \xi_{r-1,s}^{(1)} & \xi_{r-1,s}^{(2)} & \cdots & \xi_{r-1,s}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{r-p+1,s}^{(1)} & \xi_{r-p+1,s}^{(2)} & \cdots & \xi_{r-p+1,s}^{(p)} \end{vmatrix} \begin{vmatrix} \xi_{r,s}^{(1)} & \xi_{r,s}^{(2)} & \cdots & \xi_{r,s}^{(p)} \\ \xi_{r-1,s}^{(1)} & \xi_{r-1,s}^{(2)} & \cdots & \xi_{r-1,s}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{r-p+1,s}^{(1)} & \xi_{r-p+1,s}^{(2)} & \cdots & \xi_{r-p+1,s}^{(p)} \end{vmatrix}^{-1}. \quad (54)$$

*Proof.* We remark that the elements in the first row of the two matrices involved in eq. (54), have the same cofactors and therefore the cofactor expansion of the first of the above determinants, expanded along its first row, can be expressed as:

$$\begin{vmatrix} \xi_{t,s}^{(1)} & \xi_{t,s}^{(2)} & \cdots & \xi_{t,s}^{(p)} \\ \xi_{r-1,s}^{(1)} & \xi_{r-1,s}^{(2)} & \cdots & \xi_{r-1,s}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{r-p+1,s}^{(1)} & \xi_{r-p+1,s}^{(2)} & \cdots & \xi_{r-p+1,s}^{(p)} \end{vmatrix} = \xi_{t,s}^{(1)} \text{Cof}[\xi_{r,s}^{(1)}] + \xi_{t,s}^{(2)} \text{Cof}[\xi_{r,s}^{(2)}] + \cdots + \xi_{t,s}^{(p)} \text{Cof}[\xi_{r,s}^{(p)}]. \quad (55)$$

Let  $(t, r) \in \mathbb{Z}_s \times \mathbb{Z}_s$ . In view of eqs. (52) and (51) and using the well known cofactor formula of the inverse matrix  $\Xi_{r,s}^{-1}$ , we have:

$$\begin{aligned} H(t, r) &= \mathbf{e}_1 \mathbf{G}_{t,r} \mathbf{e}'_1 = \mathbf{e}_1 \Xi_{t,s} \Xi_{r,s}^{-1} \mathbf{e}'_1 \\ &= \mathbf{e}_1 \begin{bmatrix} \xi_{t,s}^{(1)} & \cdots & \xi_{t,s}^{(p)} \\ \xi_{t-1,s}^{(1)} & \cdots & \xi_{t-1,s}^{(p)} \\ \vdots & \ddots & \vdots \\ \xi_{t-p+1,s}^{(1)} & \cdots & \xi_{t-p+1,s}^{(p)} \end{bmatrix} \begin{bmatrix} \xi_{r,s}^{(1)} & \cdots & \xi_{r,s}^{(p)} \\ \xi_{r-1,s}^{(1)} & \cdots & \xi_{r-1,s}^{(p)} \\ \vdots & \ddots & \vdots \\ \xi_{r-p+1,s}^{(1)} & \cdots & \xi_{r-p+1,s}^{(p)} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= \mathbf{e}_1 \begin{bmatrix} \xi_{t,s}^{(1)} & \cdots & \xi_{t,s}^{(p)} \\ \xi_{t-1,s}^{(1)} & \cdots & \xi_{t-1,s}^{(p)} \\ \vdots & \ddots & \vdots \\ \xi_{t-p+1,s}^{(1)} & \cdots & \xi_{t-p+1,s}^{(p)} \end{bmatrix} \left( \frac{1}{|\Xi_{r,s}|} \begin{bmatrix} \text{Cof}[\xi_{r,s}^{(1)}] & \cdots & \text{Cof}[\xi_{r-p+1,s}^{(1)}] \\ \text{Cof}[\xi_{r,s}^{(2)}] & \cdots & \text{Cof}[\xi_{r-p+1,s}^{(2)}] \\ \vdots & \ddots & \vdots \\ \text{Cof}[\xi_{r,s}^{(p)}] & \cdots & \text{Cof}[\xi_{r-p+1,s}^{(p)}] \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= \frac{\mathbf{e}_1}{|\Xi_{r,s}|} \begin{bmatrix} \xi_{t,s}^{(1)} & \xi_{t,s}^{(2)} & \cdots & \xi_{t,s}^{(p)} \\ \xi_{t-1,s}^{(1)} & \xi_{t-1,s}^{(2)} & \cdots & \xi_{t-1,s}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{t-p+1,s}^{(1)} & \xi_{t-p+1,s}^{(2)} & \cdots & \xi_{t-p+1,s}^{(p)} \end{bmatrix} \begin{bmatrix} \text{Cof}[\xi_{r,s}^{(1)}] \\ \text{Cof}[\xi_{r,s}^{(2)}] \\ \vdots \\ \text{Cof}[\xi_{r,s}^{(p)}] \end{bmatrix} \\ &= \frac{[1, 0, \dots, 0]}{|\Xi_{r,s}|} \begin{bmatrix} \xi_{t,s}^{(1)} \text{Cof}[\xi_{r,s}^{(1)}] + \xi_{t,s}^{(2)} \text{Cof}[\xi_{r,s}^{(2)}] + \cdots + \xi_{t,s}^{(p)} \text{Cof}[\xi_{r,s}^{(p)}] \\ \xi_{t-1,s}^{(1)} \text{Cof}[\xi_{r,s}^{(1)}] + \xi_{t-1,s}^{(2)} \text{Cof}[\xi_{r,s}^{(2)}] + \cdots + \xi_{t-1,s}^{(p)} \text{Cof}[\xi_{r,s}^{(p)}] \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{t-p+1,s}^{(1)} \text{Cof}[\xi_{r,s}^{(1)}] + \xi_{t-p+1,s}^{(2)} \text{Cof}[\xi_{r,s}^{(2)}] + \cdots + \xi_{t-p+1,s}^{(p)} \text{Cof}[\xi_{r,s}^{(p)}] \end{bmatrix} \\ &= |\Xi_{r,s}|^{-1} (\xi_{t,s}^{(1)} \text{Cof}[\xi_{r,s}^{(1)}] + \xi_{t,s}^{(2)} \text{Cof}[\xi_{r,s}^{(2)}] + \cdots + \xi_{t,s}^{(p)} \text{Cof}[\xi_{r,s}^{(p)}]). \end{aligned}$$

Comparing eq. (55) with the above last expression of  $H(t, r)$  for  $(t, r) \in \mathbb{Z}_s \times \mathbb{Z}_s$ , eq. (54) follows. Moreover, for any  $m \in \llbracket 1, p \rrbracket$ ,  $H(s+1-m, r)$  in eq. (54) is well defined on the extended domain  $\mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ .  $\square$

An equivalent form of  $H(t, r)$  for  $s+1-p \leq t \leq s$  is derived below

$$\begin{aligned} H(s+1-m, r) &= \frac{\xi_{s+1-m,s}^{(1)} \text{Cof}[\xi_{r,s}^{(1)}] + \xi_{s+1-m,s}^{(2)} \text{Cof}[\xi_{r,s}^{(2)}] + \cdots + \xi_{s+1-m,s}^{(p)} \text{Cof}[\xi_{r,s}^{(p)}]}{|\Xi_{r,s}|} \\ &\quad (\text{as } \xi_{s+1-m,s}^{(i)} = 1, \text{ if } i = m, \text{ or } 0, \text{ if } i \neq m) \\ &= \frac{\text{Cof}[\xi_{r,s}^{(m)}]}{|\Xi_{r,s}|} = \frac{(-1)^{m+1}}{|\Xi_{r,s}|} \begin{vmatrix} \xi_{r-1,s}^{(1)} & \cdots & \xi_{r-1,s}^{(m-1)} & \xi_{r-1,s}^{(m+1)} & \cdots & \xi_{r-1,s}^{(p)} \\ \vdots & & \vdots & \vdots & & \vdots \\ \xi_{r-p+1,s}^{(1)} & \cdots & \xi_{r-p+1,s}^{(m-1)} & \xi_{r-p+1,s}^{(m+1)} & \cdots & \xi_{r-p+1,s}^{(p)} \end{vmatrix}, \end{aligned}$$

provided that if  $m = 1$  (resp.  $m = p$ ), then the first two (resp. last two) columns illustrated in the latter determinant expression above are vanished.

Taking into account that  $\{\xi_{\cdot, s}^{(m)}\}_{1 \leq m \leq p}$  are fundamental solutions defined for  $t \in \mathbb{Z}_{s+1-p}$  (see Theorem 2), some crucial values of  $H(t, r)$  on  $\mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$  are verified below (see Theorem 5 and the discussion below Corollary 2).

In the following Theorem, we further enlarge the domain  $\mathfrak{Y}_s$  of Lemma 2, showing that the Green's function  $H(t, r)$  coincides with the principal determinant function  $\xi_{t, r}$  on an extended domain  $\mathcal{Z}$  defined as follows: Let  $p > 1$  and  $\mathfrak{J}_i = \{(s-i, s), (s-i+1, s+1), \dots, (t, t+i), \dots\}$  for  $1 \leq i \leq p-1$ . Let us call  $\mathfrak{J} = \bigcup_{i=1}^{p-1} \mathfrak{J}_i$  and  $\mathcal{Z} = \mathfrak{Y}_s \cup \mathfrak{J}$ . Notice that if  $p = 1$ , then  $\mathfrak{J} = \emptyset$  and  $\mathcal{Z} = \mathfrak{Y}_s$ . In what follows we assume that  $p \geq 2$ . Formally,  $(t, r) \in \mathfrak{J}$  if and only if (iff for short) there exists some  $i \in \llbracket 1, p-1 \rrbracket$  such that  $r-t = i$  (or  $r = t+i$ ). Equivalently  $(t, t+i) \in \mathfrak{J}$  iff  $i \in \llbracket 1, p-1 \rrbracket$  and  $t \in \mathbb{Z}_{s-i}$ . As  $t \geq s-i$ , it follows that  $t+i \geq s$ , whence  $(t+i) \in \mathbb{Z}_s$ .

**Lemma 3.** *The following statements hold:*

- i)  $\mathfrak{J}_i \cap \mathfrak{J}_k = \emptyset$ , whenever  $i \neq k$ .
- ii)  $\mathcal{Z} \subset \mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ .
- iii)  $\mathfrak{Y}_s \cap \mathfrak{J} = \emptyset$ , that is  $\mathfrak{Y}_s, \mathfrak{J}$  are disjoint sets.
- iv)  $\mathfrak{J} = \mathcal{Z} \setminus \mathfrak{Y}_s$ .
- v)  $(t, r) \in \mathcal{Z}$  iff  $t \in \mathbb{Z}_{s+1-p}$  and  $r \in \llbracket s, t-p+1 \rrbracket$ .

*Proof.* i) Let  $(t, r) \in \mathfrak{J}_i$ . Then  $r-t = i \neq k$ , whence  $(t, r) \notin \mathfrak{J}_k$  and the result follows.

ii) By Definition we have:  $\mathfrak{Y}_s \subset \mathbb{Z}_s \times \mathbb{Z}_s \subset \mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ . Also  $(t, t+i) \in \mathfrak{J}$  iff  $t \in \mathbb{Z}_{s-i}$  and  $i \in \llbracket 1, p-1 \rrbracket$ . As  $\mathbb{Z}_{s-i} \subset \mathbb{Z}_{s+1-p}$ , it follows that  $t \in \mathbb{Z}_{s+1-p}$ . Moreover, as  $(t+i) \in \mathbb{Z}_s$ , it follows that  $(t, t+i) \in \mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ , whence  $\mathfrak{J} \subset \mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ . As both  $\mathfrak{Y}_s$  and  $\mathfrak{J}$  are subsets of  $\mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ , we conclude that  $\mathcal{Z} = \mathfrak{Y}_s \cup \mathfrak{J} \subset \mathbb{Z}_{s+1-p} \times \mathbb{Z}_s$ .

iii) Let  $(t, r) \in \mathfrak{Y}_s$ . Then as  $p \geq 2$  we have:  $s \leq r \leq t < t+i$  for any  $i \in \llbracket 1, p-1 \rrbracket$ . Accordingly  $r \neq t+i$  for all  $i \in \llbracket 1, p-1 \rrbracket$ . Thus  $(t, r) \notin \mathfrak{J}_i$  for all  $i \in \llbracket 1, p-1 \rrbracket$  and so  $(t, r) \notin \bigcup_{i=1}^{p-1} \mathfrak{J}_i$ . The latter implies that  $\mathfrak{Y}_s \cap (\bigcup_{i=1}^{p-1} \mathfrak{J}_i) = \emptyset$  and the assertion follows from the Definition  $\mathfrak{J} = \bigcup_{i=1}^{p-1} \mathfrak{J}_i$ .

iv) As  $\mathcal{Z} = \mathfrak{Y}_s \cup \mathfrak{J}$  and  $\mathfrak{Y}_s \cap \mathfrak{J} = \emptyset$ , the assertion follows.

v) In what follows we shall use the statements (a) to (c) below:

- a)  $t \in \mathbb{Z}_{s-i}$  iff  $(t \geq s-i$  and  $t \in \mathbb{Z}_{s+1-p})$  iff  $(t \in \mathbb{Z}_{s+1-p}$  and  $s \leq t+i)$  (notice that  $t \in \mathbb{Z}_{s+1-p}$  is redundant).
- b)  $(r = t+i$  and  $1 \leq i \leq p-1)$  iff  $t+1 \leq r \leq t+1-p$ .
- c)  $[(s \leq r$  and  $t+1 \leq r \leq t+1-p)$  or  $s \leq r \leq t]$  iff  $s \leq r \leq t+1-p$ .

The Definition of  $\mathfrak{J}$  followed by statements (a) and (b) imply:  $(t, r) \in \mathfrak{J}$  iff  $(t \in \mathbb{Z}_{s-i}$  and  $1 \leq i \leq p-1$  and  $r = t+i)$  iff  $(t \in \mathbb{Z}_{s+1-p}$  and  $s \leq t+i = r$  and  $t+1 \leq r \leq t+1-p)$ . Taking into account that  $(t, r) \in \mathfrak{Y}_s$  iff  $t \in \mathbb{Z}_s$  and  $s \leq r \leq t$ , it follows from (c) that:  $(t, r) \in \mathfrak{J} \cup \mathfrak{Y}_s$  iff  $(t \in \mathbb{Z}_{s+1-p}$  and  $s \leq r$  and  $t+1 \leq r \leq t+1-p)$  or  $(t \in \mathbb{Z}_s$  and  $s \leq r \leq t)$  iff  $(t \in \mathbb{Z}_{s+1-p}$  and  $s \leq r \leq t+1-p)$ , as asserted.  $\square$

**Theorem 5.** *Let  $H(t, r)|_{\mathcal{Z}}$  be the restriction of the Green's function to  $\mathcal{Z}$ . Then  $H(t, r)|_{\mathcal{Z}} = \xi_{t, r}$ .*

*Proof.* The Definition in eq. (30), applied for  $m = 1$ , implies that  $\xi_{t, r} = 0$ , whenever  $r > t$  (or  $\xi_{t, t+j} = 0$ , whenever  $j \geq 1$ ) for all  $t \in \mathbb{Z}_{s+1-p}$ . In view of Lemma 2, it suffices to show that  $H(t, r) = 0$  on the set  $\mathcal{Z} \setminus \mathfrak{Y}_s$ , and therefore on account Lemma 3 (iv), it suffices to show that  $H(t, r) = 0$  on the set  $\mathfrak{J}$ . Now, the result follows from the fact that for any  $i \in \llbracket 1, p-1 \rrbracket$  and any  $t \in \mathbb{Z}_{s-i}$ , the numerator of  $H(t, t+i)$  in eq. (54) is zero, that is

$$\begin{vmatrix} \xi_{t, s}^{(1)} & \xi_{t, s}^{(2)} & \cdots & \xi_{t, s}^{(p)} \\ \xi_{t+i-1, s}^{(1)} & \xi_{t+i-1, s}^{(2)} & \cdots & \xi_{t+i-1, s}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{t+i-p+1, s}^{(1)} & \xi_{t+i-p+1, s}^{(2)} & \cdots & \xi_{t+i-p+1, s}^{(p)} \end{vmatrix} = 0,$$

since its first row coincides with one of its remaining rows for any  $i = 1, 2, \dots, p-1$ , while its denominator is nonzero, i.e.,  $|\Xi_{t+i, s}| \neq 0$ , since  $t+i \geq s$ , whence  $\Xi_{t+i, s}$  is invertible (see Corollary 1).  $\square$

**Corollary 2.** *The general solution of eq. (28) (or the general homogeneous solution of eq. (25)) can be explicitly expressed in terms of the Green's function, the varying coefficients and the sequence of prescribed values  $\{y_{r+1-m}\}_{1 \leq m \leq p}$  as:*

$$y_t = \sum_{m=1}^p \sum_{i=1}^{p+1-m} \phi_{m-1+i}(r+i) H(t, r+i) y_{r+1-m} \quad \text{for all } t \geq r+1. \quad (56)$$

*Proof.* The range of values of  $i$  in the second sum of eq. (43) is:  $1 \leq i \leq p+1-m$ . Taking into account that  $s+1 \leq r+1 \leq t$  and  $1 \leq i \leq p+1-m \leq p$  (since  $m \geq 1$ ), the following chain of inequalities holds:

$$s \leq s+1 \leq r+1 \leq r+i \leq r+p+1-m \leq r+p = (r+1) - 1 + p \leq t-1+p. \quad (57)$$

Thus  $s \leq r+i \leq t-1+p$  for any  $i \in \llbracket 1, p-1 \rrbracket$  and therefore for all  $i$  such that  $1 \leq i \leq p+1-m$ . We infer that  $t \in \mathbb{Z}_{s+1-p}$  and  $s \leq r+i \leq t-1+p$ . Thus, Lemma 3(v) implies that  $(t, r+i) \in \mathcal{Z}$  and Theorem 5 allows us to replace  $\xi_{t, r+i}$  in eq. (43) with  $H(t, r+i)$ , which, in turn, implies that eq. (56) holds true, as claimed.  $\square$







$$\det(\mathbf{H}_k) = h_{k,1} + \sum_{j=2}^k \sum_{k_1=j}^k \sum_{k_2=j-1}^{k_1-1} \cdots \sum_{k_{j-1}=2}^{k_{j-2}-1} h_{k,k_1} \prod_{m=2}^{j-1} h_{k_{m-1}-1, k_m} h_{k_{j-1}-1, 1}. \quad (66)$$

Applying the assignment in eq. (64) to eq. (66), the nested sum representation of  $H(t, r)|_{\mathcal{Z}}$  (or  $\xi_{t,r}$ ) takes the form:

$$H(t, r)|_{\mathcal{Z}} = \xi_{t,r} \quad (67)$$

$$= \begin{cases} \phi_{t-r}(t) + \sum_{j=2}^{t-r} \sum_{k_1=j}^{t-r} \sum_{k_2=j-1}^{k_1-1} \cdots \sum_{k_{j-1}=2}^{k_{j-2}-1} \phi_{t-r-k_{j-1}+1}(t) \prod_{m=2}^{j-1} \phi_{k_{m-1}-k_m}(r+k_{m-1}-1) \phi_{k_{j-1}-1}(r+k_{j-1}-1), & \text{if } s \leq r < t \\ 1, & \text{if } t = r \\ 0, & \text{elsewhere} \end{cases}$$

Proceeding with the above mentioned assignment, the Green's function restriction  $H(t, r)|_{\mathcal{Z}}$  can also be represented by Mallik's combinatorial formula in [4], as adjusted for Hessenbergians in [6] (see eq. (9) therein).

### 6.3. Companion Matrix Product

By virtue of Theorem 5, it follows from the chain of inequalities in eq. (57) that we can replace  $\xi_{t,r+j}$  with the Green's function restriction  $H(t, r+j)|_{\mathcal{Z}}$  in eq. (42) for  $1 \leq m \leq p$  to obtain the expressions

$$\xi_{t,r}^{(m)} = \begin{cases} \sum_{j=1}^{p+1-m} \phi_{m-1+j}(r+j) H(t, r+j), & 2 \leq m \leq p, \\ H(t, r), & m = 1, \end{cases} \quad t \geq r+1, \quad (68)$$

noticing that if  $t \leq r$ , then the corresponding values of  $\xi_{t,r}^{(m)}$  are given by eq. (32). Therefore the elements of the companion matrix product can be expressed directly in terms of the Green's function. Applying eq. (65) to (68), we conclude that  $\xi_{t,r}^{(m)}$  are equipped with the following Leibnizian representations:

$$\xi_{t,r}^{(m)} = \begin{cases} \sum_{j=1}^{p-m+1} \phi_{m-1+j}(r+j) \sum_{q=0}^{2^{t-r-1-j}-1} \prod_{i=1}^{t-r-j} \phi_{i-\sigma_{t-r-j,i}(q)+1}(r+j+i), & 2 \leq m \leq p, \\ \sum_{m=0}^{2^{t-r-1}-1} \prod_{i=1}^{t-r} \phi_{i-\sigma_{t-r,i}(m)+1}(r+i), & m = 1, \end{cases} \quad t \geq r+1. \quad (69)$$

As  $\mathbf{F}_{t,r} = \Xi_{t,r}$  whenever  $t \geq r \geq s$  (see Theorem 3), the expressions in eq. (69) yield compact representations for the elements of the companion matrix product in eq. (46), respectively. The latter result is to be compared with Theorem 2.1. in ([5]).

### 6.4. Green's Function Determinant-Ratio Formula

Applying the expression in eq. (69) for  $r = s$  to the right-hand side determinant elements of eq. (54), we establish compact representations for the elements of the Green's function determinant ratio formula in eq. (54), formulated by Leibnizian representations.

Similarly, applying the expression in eq. (67) for  $r = s$  to eq. (68), we obtain nested sum representations for the elements of Green's function determinant ratio formula in eq. (54).

### 6.5. Leibnizian Solution Representation

Given any sequence of prescribed values  $\{y_{r+1-p}, \dots, y_r\}$  for  $r \geq s$  fixed, the expressions in eq. (65) applied to eq. (62) yield the compact Leibnizian representation of the solution to eq. (25) for  $t \geq r+1$ , that is:

$$y_t = \sum_{m=1}^p y_{r+1-m} \sum_{j=1}^{p+1-m} \phi_{m-1+j}(r+j) \sum_{q=0}^{2^{t-r-j-1}-1} \prod_{i=1}^{t-r-j} \phi_{i-\sigma_{t-r-j,i}(q)+1}(r+j+i) \\ + \sum_{j=1}^{t-r} v_{r+j} \sum_{q=0}^{2^{t-r-j-1}-1} \prod_{i=1}^{t-r-j} \phi_{i-\sigma_{t-r-j,i}(q)+1}(r+j+i). \quad (70)$$



Eq. (70) is to be compared with the nested sum representation of the general solution established in [6] (see eq. (17) therein). In Appendix C, both Algorithms 1 and 2 are applied along with the assignment in eq. (64) to verify the compact solution representation in eq. (70), evaluated for any given sequence of prescribed values.

## 7. Future Work

The results of this work can be extended in multiple directions. We highlight two of them:

Our results can be extended to cover an explicit representation to the solution of infinite order linear difference equations with constant or variable coefficients (ILDEs). This can be established by extending our results to cover linear difference equations of unbounded order, but of finite kernel index  $p$  (ULDE( $p$ )) (see [20]).<sup>5</sup> An ULDE( $p$ ) is naturally derived as a  $p$  order truncation of the ILDE, yielding an approximation of  $p$  order to the original ILDE. The fundamental solution set obtained here can be similarly formulated considering full lower Hessenberg matrices in place of banded ones, as in eq. (39) of Lemma 1. As a consequence, the general solution of ULDEs( $p$ ) is given by eq. (60), using full Hessenbergians in place of  $\xi_{t,r+i}$ . In a similar manner the Leibnizian and nested sum representations of the solution can be directly derived. The corresponding solution of the ILDE turns out to be the limit of the associated ULDEs( $p$ ), as  $p \rightarrow \infty$ .

Our methodology can be generalized to the case of multivariate VC-LDEs( $p$ ), where square matrices with elements variable coefficients are used in place of scalar valued variable coefficients, as in eq. (25). Working on the algebra of noncommutative rings, an explicit form to the general solution representation of multi-variate difference equations with variable matrix coefficients is obtained. This result has some remarkable consequences on the fundamental properties of multivariate ARMA models.

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<sup>5</sup>The term LDEs of ascending order of index  $N$  is equivalently used there for the ULDEs( $p$ ).

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## Appendices

In Appendix A, we provide proofs for the results reported in the main body of the paper. The infinite Gaussian elimination algorithm is presented in Appendix B. Finally, in Appendix C, we provide two algorithms translated into automatically executable computer programs. The first, constructs and verifies the Leibnizian compact representation of Hessenbergians in eq. (24). The second, constructs the Leibnizian representation of the Green’s function  $H(t, r)$  on  $\mathcal{Z}$ , followed by the corresponding representation of the general solution of a VC-LDE( $p$ ).

### Appendix A [Proofs]

**Proposition A1.** *The following statements hold:*

- i) The recurrence in eq. (4) can be equivalently expressed by eq. (6).*
- ii) The number of non-trivial SEPs of Hessenbergians is  $2^{k-1}$ , that is  $\text{card}(\mathcal{E}_k) = 2^{k-1}$ .*

*Proof.* *i)* Applying the assignments  $h_{i,j} = c_{i,j}$ , whenever  $j \neq i + 1$  and  $h_{i,i+1} = -c_{i,i+1}$  to (4) after some algebraic manipulations, demonstrated below, eq. (6) follows:

$$\begin{aligned}
\det(\mathbf{H}_k) &= h_{k,k} \det(\mathbf{H}_{k-1}) + \sum_{i=1}^{k-1} (-1)^{k-i} h_{k,i} \prod_{j=i}^{k-1} h_{j,j+1} \det(\mathbf{H}_{i-1}) \\
&= c_{k,k} \det(\mathbf{H}_{k-1}) + \sum_{i=1}^{k-1} (-1)^{k-i} c_{k,i} \prod_{j=i}^{k-1} (-1) c_{j,j+1} \det(\mathbf{H}_{i-1}) \\
&= c_{k,k} \det(\mathbf{H}_{k-1}) + \sum_{i=1}^{k-1} (-1)^{k-i} c_{k,i} (-1)^{k-i} \prod_{j=i}^{k-1} c_{j,j+1} \det(\mathbf{H}_{i-1}) \\
&= c_{k,k} \det(\mathbf{H}_{k-1}) + \sum_{i=1}^{k-1} (-1)^{2(k-1)} \prod_{j=i}^{k-1} c_{k,i} c_{j,j+1} \det(\mathbf{H}_{i-1}) \\
&= c_{k,k} \det(\mathbf{H}_{k-1}) + \sum_{i=1}^{k-1} \prod_{j=i}^{k-1} c_{k,i} c_{j,j+1} \det(\mathbf{H}_{i-1}).
\end{aligned}$$

ii) Let  $n(r)$  be the number of distinct non-trivial SEPs associated with  $\mathbf{H}_r$ . Taking into account that  $n(0) = n(1) = 1$ , the recurrence in eq. (7) implies that  $n(r) = n(0) + n(1) + \sum_{i=2}^{r-1} n(i)$  for all  $r \geq 1$ , provided that  $\sum_{i=j}^l a(i) = 0$ , whenever  $j > i$ . This can be rewritten as:

$$n(r) = 1 + \sum_{i=1}^{r-1} n(i) \quad \text{for all } r \geq 1. \quad (\text{A.1})$$

Working with the weak induction on  $k \geq 1$  we shall show that  $n(k) = 2^{k-1}$  for all  $k \geq 1$ . The basis step  $n(1) = 2^0 = 1$  holds true. The induction hypothesis assumes that the statement  $n(k-1) = 2^{k-2}$  holds true. Starting with eq. (A.1), applied for  $r = k$ , we obtain:

$$\begin{aligned} n(k) &= 1 + \sum_{i=1}^{k-1} n(i) \\ (\text{equivalently}) &= (1 + \sum_{i=1}^{k-2} n(i)) + n(k-1) \\ (\text{apply eq. (A.1) for } r = k-1) &= n(k-1) + n(k-1) \\ (\text{equivalently}) &= 2 \cdot n(k-1) \\ (\text{by the induction Hypothesis}) &= 2 \cdot 2^{k-2} \\ (\text{equivalently}) &= 2^{k-1} \end{aligned}$$

This satisfies the induction step, and the proof is completed.  $\square$

**Proposition A2.** *The standard IS, say  $c_{i,j}$ , of any initial string  $C[i-1; \ell]$  is uniquely determined by the number, say  $m$  ( $0 \leq m \leq i-1$ ), of consecutive non-standard predecessors of  $c_{i,j}$  and in this case  $j = i - m$ .*

*Proof.* The hypothesis entails that the initial string can be expressed as:

$$C[i-1; \ell] = c_{1\ell_1} \dots c_{i-m-2, \ell_{i-m-2}} \underbrace{c_{i-m-1, \ell_{i-m-1}}}_{\text{standard}} \underbrace{c_{i-m, i-m+1} c_{i-m+1, i-m+2} \dots c_{i-1, i}}_{m \text{ non-standard factors}}.$$

As  $c_{i,j}$  is a standard IS of  $C[i-1; \ell]$ , we can write  $j = i - n$  for some  $n = 0, 1, 2, \dots, i-1$ . In order to show that this standard IS of  $C[i-1; \ell]$  is  $c_{i, i-m}$  (or  $n = m$ ), it suffices to show that none of the factors of  $C[i-1; \ell]$  has column index  $i - m$ . First, the non-standard factors next to  $c_{i-m-1, \ell_{i-m-1}}$  have column indices  $i - m + 1, \dots, i$ . Thus  $(i - m) \notin \{i - m + 1, \dots, i\}$ . Moreover, as  $c_{i-m-1, \ell_{i-m-1}}$  is standard, we infer that  $\ell_{i-m-1} \neq i - m$ , since otherwise  $c_{i-m-1, \ell_{i-m-1}} = c_{i-m-1, i-m}$  which is non-standard. Finally if  $\ell_{i-m-2} = i - m$ , then  $c_{i-m-2, \ell_{i-m-2}} = c_{i-m-2, i-m}$ , which is a trivial entry, since  $i - m - (i - m - 2) = 2$ . The same holds for all the preceding factors of  $c_{i-m-2, \ell_{i-m-2}}$  and the result follows.  $\square$

**Proposition A3.** *The function  $f_k : \mathcal{E}_k \mapsto \mathfrak{R}_k$  defined in eq. (2.2) is bijective.*

*Proof.* As the set  $\mathfrak{R}_k$  and the set  $\mathcal{E}_k$  have the same number of elements ( $2^{k-1}$ ) it suffices to show that  $f_k$  is injective. Let us consider  $Q = c_{1, \ell_1} c_{2, \ell_2} \dots c_{k, \ell_k}$  and  $P = c_{1, l_1} c_{2, l_2} \dots c_{k, l_k}$  in  $\mathcal{E}_k$  such that  $f_k(Q) = f_k(P)$ . We need to show that  $Q = P$  or equivalently that  $\ell = l$ . Let us call  $f_k(Q) = f_k(P) = \mathbf{r}$ , where  $\mathbf{r} = (r_1, r_2, \dots, r_{k-1}, 1)$ . We examine the following cases:

- I) Let  $r_i = 0$ . The Definition of  $f_k$  implies that the  $i$ th non-trivial factor of  $C$  and  $P$  is non-standard. As there is only one such factor, that is the entry  $(i, i+1)$ , it must be the factor  $c_{i, i+1}$ . Thus  $\ell_i = l_i = i + 1$ .
- II) Let  $r_i = 1$ . The Definition of  $f_k$  implies that the  $i$ th non-trivial factors of  $Q$  and  $P$ , say  $c_{i, \ell_i}$  and  $c_{i, l_i}$ , are standard. Property 4 in Proposition 1, entails that  $c_{i, \ell_i}$  and  $c_{i, l_i}$  are completely determined by the number of the consecutive non-standard predecessors of  $c_{i, \ell_i}$  and  $c_{i, l_i}$ . The result follows from case I, which entails that both SEPs have identical non-standard factors occupying the same order positions.

Therefore in all cases  $\ell_i = l_i$ , whence  $C = P$  as required.  $\square$

**Proposition A4.** *The function  $\zeta_{k,i}(\mathbf{r})$  in defined in (11) can be expressed as an elementary integer function, which is given by :*

$$\zeta_{k,i}(\mathbf{r}) = r_i(i - \max_{0 \leq j < i} \{j \cdot r_j\}) - 1 \quad (\text{A.2})$$

*Proof.* Let us call  $z_{k,i}(\mathbf{r}) = r_i(i - \max_{0 \leq j < i} \{j \cdot r_j\}) - 1$ , while  $\zeta_{k,i}(\mathbf{r})$  is given by eq. (11). We shall show that  $z_{k,i}(\mathbf{r}) = \zeta_{k,i}(\mathbf{r})$  for all  $\mathbf{r} = (r_1, r_2, \dots, r_i, \dots, r_{k-1}, 1) \in \mathfrak{R}_k$ . First we notice that if  $i = 1$ , then, in view of eq. (13),

the equality  $z_{k,1}(\mathbf{r}) = \zeta_{k,1}(\mathbf{r})$  holds true for all  $\mathbf{r} \in \mathfrak{R}_k$ . It remains to show that  $z_{k,i}(\mathbf{r}) = \zeta_{k,i}(\mathbf{r})$  for all  $i \geq 2$ . In this case we have:

$$\max_{0 \leq j < i} \{j \cdot r_j\} = \max\{0 \cdot r_0, 1 \cdot r_1, \dots, (i-1) \cdot r_{i-1}\} = \max\{1 \cdot r_1, \dots, (i-1) \cdot r_{i-1}\} = \max_{1 \leq j < i} \{j \cdot r_j\}.$$

Therefore in the case when  $i \geq 2$ , we can use the expression:  $z_{k,i}(\mathbf{r}) = r_i(i - \max_{1 \leq j < i} \{j \cdot r_j\}) - 1$ . We examine the following cases:

i) Let  $r_i = 0$ . Then a simple evaluation gives  $z_{k,i}(\mathbf{r}) = -1 = \zeta_{k,i}(\mathbf{r})$ .

ii) Let  $r_i = 1$ . We examine the following sub-cases:

a) Let  $\max_{1 \leq j < i} \{j \cdot r_j\} = 0$ . Then  $r_j = 0$  for all  $j$  such that  $1 \leq j < i$ . Applying the formula of  $z_{k,i}(\mathbf{r})$  we get

$$z_{k,i}(\mathbf{r}) = i - 1 = \zeta_{k,i}(\mathbf{r}).$$

b) Let  $\max_{1 \leq j < i} \{j \cdot r_j\} = M$  and  $M > 0$ . Then we can write:

$$\{j \cdot r_j\}_{1 \leq j < i} = \{1 \cdot r_1, \dots, (M-1)r_{M-1}, Mr_M, (M+1)r_{M+1}, \dots, (i-1)r_{i-1}\}.$$

We shall show that  $r_M = 1$ . On the contrary we assume that  $r_M = 0$ . Then the following equality must hold

$$M = \max\{1r_1, \dots, (M-1)r_{M-1}, 0, (M+1)r_{M+1}, \dots, (i-1)r_{i-1}\},$$

which is contradictory, because  $M \notin \{1, 2, \dots, M-1, 0, M+1, \dots, i-1\}$ .

In this case we further conclude that  $r_{M+1} = r_{M+2} = \dots = r_{i-1} = 0$ ; for if otherwise  $\max_{1 \leq j < i} \{j \cdot r_j\} > M$ .

Therefore we conclude that:  $\{j \cdot r_j\}_{1 \leq j \leq i} = \{1r_1, \dots, Mr_M, 0\}$ . As the number of consecutive 0s between  $r_M = 1$  and  $r_i = 1$  is  $i - M - 1$ , Definition (11) gives  $\zeta_{k,i}(\mathbf{r}) = i - M - 1$ . Also the formula of  $z_{k,i}(\mathbf{r})$  yields  $z_{k,i}(\mathbf{r}) = r_i(i - M) - 1 = 1(i - M) - 1 = i - M - 1$ , whence  $z_{k,i}(\mathbf{r}) = \zeta_{k,i}(\mathbf{r})$ .

The proof of Proposition is complete.  $\square$

**Proposition A5.** Let  $n, d \in \mathbb{Z}$  and  $d \geq 1$ . The following identity of nested divisions holds:

$$\underbrace{[\dots [ [n : d] : d ] \dots ] : d}_m = [n : d^m]. \quad (\text{A.3})$$

*Proof.* Let  $x$  be a real number and  $p, q$  be positive integers. We shall use the well known identity

$$[\![x] : p] = [x : p]. \quad (\text{A.4})$$

(see [16], eq. (P15), p. 47).

Taking into account that  $(x : q) : p = x : (p \cdot q)$ , it follows from (A.4) that:

$$[\![x : q] : p] = [(x : q) : p] = [x : (p \cdot q)]. \quad (\text{A.5})$$

To verify (A.3) we use induction on  $m \in \mathbb{Z}_0$ . Clearly, the identity holds for  $m = 0$ . Let us assume that the identity (A.3) holds for  $m = i$ , that is:

$$\underbrace{[\dots [ [n : d] : d ] \dots ] : d}_i = [n : d^i].$$

The induction hypothesis combined with (A.5) implies:

$$[\underbrace{[\dots [ [n : d] : d ] \dots ] : d}_{[n : d^i]}] : d = [[n : d^i] : d] = [n : (d^i \cdot d)] = [n : d^{i+1}].$$

This completes the induction.  $\square$

**Proposition A6.** The following equality of sets holds:

$$\mathbb{Z}_{s+1-p} = \bigcup_{j=1}^{p-1} \mathbb{Z}_{s-j}.$$

*Proof.* As  $\mathbb{Z}_{s-j} \subseteq \mathbb{Z}_{s-(p-1)}$  for all  $j \in \llbracket 1, p-1 \rrbracket$ , it follows that  $\mathbb{Z}_{s-(p-1)} = \bigcup_{j=1}^{p-1} \mathbb{Z}_{s-j}$ . Now the equality follows from

$$\mathbb{Z}_{s+1-p} = \mathbb{Z}_{s-(p-1)} = \bigcup_{j=1}^{p-1} \mathbb{Z}_{s-j},$$

as asserted  $\square$

**Proposition A7.** The Casoratian  $|\Xi_{t,r}|$  defined in eq. (35) satisfies the first order linear difference equation:

$$|\Xi_{t,r}| = (-1)^{p-1} \phi_p(t) |\Xi_{t-1,r}|. \quad (\text{A.6})$$

*Proof.* If we replace the elements  $\xi_{t,r}^{(m)}$  for  $1 \leq m \leq p$  in the first row of  $|\Xi_{t,r}|$  with the right-hand side of the recurrence (41)  $|\Xi_{t,r}|$  takes the form:

$$|\Xi_{t,r}| = \begin{vmatrix} \phi_p(t)\xi_{t-p,r}^{(1)} + \dots + \phi_1(t)\xi_{t-1,r}^{(1)} & \dots & \phi_p(t)\xi_{t-p,r}^{(p)} + \dots + \phi_1(t)\xi_{t-1,r}^{(p)} \\ \xi_{t-1,r}^{(1)} & \dots & \xi_{t-1,r}^{(p)} \\ \vdots & \ddots & \vdots \\ \xi_{t-p+1,r}^{(1)} & \dots & \xi_{t-p+1,r}^{(p)} \end{vmatrix}. \quad (\text{A.7})$$

Using the multi-linearity of determinants in rows, eq. (A.7) can be written as

$$|\Xi_{t,r}| = \phi_p(t) \begin{vmatrix} \xi_{t-p,r}^{(1)} & \dots & \xi_{t-p,r}^{(p)} \\ \xi_{t-1,r}^{(1)} & \dots & \xi_{t-1,r}^{(p)} \\ \vdots & \ddots & \vdots \\ \xi_{t-p+1,r}^{(1)} & \dots & \xi_{t-p+1,r}^{(p)} \end{vmatrix} + \phi_{p-1}(t) \begin{vmatrix} \xi_{t-p+1,r}^{(1)} & \dots & \xi_{t-p+1,r}^{(p)} \\ \xi_{t-1,r}^{(1)} & \dots & \xi_{t-1,r}^{(p)} \\ \vdots & \ddots & \vdots \\ \xi_{t-p+1,r}^{(1)} & \dots & \xi_{t-p+1,r}^{(p)} \end{vmatrix} \\ + \dots + \phi_1(t) \begin{vmatrix} \xi_{t-1,r}^{(1)} & \dots & \xi_{t-1,r}^{(p)} \\ \xi_{t-1,r}^{(1)} & \dots & \xi_{t-1,r}^{(p)} \\ \vdots & \ddots & \vdots \\ \xi_{t-p+1,r}^{(1)} & \dots & \xi_{t-p+1,r}^{(p)} \end{vmatrix}.$$

The values of the determinants from the second term up to and including the last term of the right-hand side of the above equality are zero, since they have two identical rows, whence

$$|\Xi_{t,r}| = \phi_p(t) \begin{vmatrix} \xi_{t-p,r}^{(1)} & \xi_{t-p,r}^{(2)} & \dots & \xi_{t-p,r}^{(p)} \\ \xi_{t-1,r}^{(1)} & \xi_{t-1,r}^{(2)} & \dots & \xi_{t-1,r}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{t-p+1,r}^{(1)} & \xi_{t-p+1,r}^{(2)} & \dots & \xi_{t-p+1,r}^{(p)} \end{vmatrix}.$$

One needs  $(p-1)$  successive row interchanges to move the first row to the last row position and the above equality can be written as

$$|\Xi_{t,r}| = (-1)^{p-1} \phi_p(t) \begin{vmatrix} \xi_{t-1,r}^{(1)} & \xi_{t-1,r}^{(2)} & \dots & \xi_{t-1,r}^{(p)} \\ \xi_{t-2,r}^{(1)} & \xi_{t-2,r}^{(2)} & \dots & \xi_{t-2,r}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{t-p+1,r}^{(1)} & \xi_{t-p+1,r}^{(2)} & \dots & \xi_{t-p+1,r}^{(p)} \\ \xi_{t-p,r}^{(1)} & \xi_{t-p,r}^{(2)} & \dots & \xi_{t-p,r}^{(p)} \end{vmatrix} = (-1)^{p-1} \phi_p(t) |\Xi_{t-1,r}|.$$

This completes the proof of Proposition.  $\square$

**Proposition A8.** The Green's function associated with the difference operator in eq. (29) has the property:

$$H(t, t+p) = \frac{1}{\phi_p(t+p)} \quad \text{for } t \in \mathbb{Z}_{s+1-p}.$$

*Proof.* Eq. (54) applied for  $r = t+p$  yields:

$$H(t, t+p) = \begin{vmatrix} \xi_{t,s}^{(1)} & \xi_{t,s}^{(2)} & \dots & \xi_{t,s}^{(p)} \\ \xi_{t+p-1,s}^{(1)} & \xi_{t+p-1,s}^{(2)} & \dots & \xi_{t+p-1,s}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{t+1,s}^{(1)} & \xi_{t+1,s}^{(2)} & \dots & \xi_{t+1,s}^{(p)} \end{vmatrix} \begin{vmatrix} \xi_{t+p,s}^{(1)} & \xi_{t+p,s}^{(2)} & \dots & \xi_{t+p,s}^{(p)} \\ \xi_{t+p-1,s}^{(1)} & \xi_{t+p-1,s}^{(2)} & \dots & \xi_{t+p-1,s}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{t+1,s}^{(1)} & \xi_{t+1,s}^{(2)} & \dots & \xi_{t+1,s}^{(p)} \end{vmatrix}^{-1}.$$

Applying  $(p-1)$  successive row interchanges to the numerator determinant of  $H(t, t+p)$  of the above equality, its first row is moved to occupy the last row position, whence:

$$H(t, t+p) = (-1)^{p-1} \frac{\begin{vmatrix} \xi_{t+p-1,s}^{(1)} & \xi_{t+p-1,s}^{(2)} & \cdots & \xi_{t+p-1,s}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{t+1,s}^{(1)} & \xi_{t+1,s}^{(2)} & \cdots & \xi_{t+1,s}^{(p)} \\ \xi_{t,s}^{(1)} & \xi_{t,s}^{(2)} & \cdots & \xi_{t,s}^{(p)} \end{vmatrix}}{\begin{vmatrix} \xi_{t+p,s}^{(1)} & \xi_{t+p,s}^{(2)} & \cdots & \xi_{t+p,s}^{(p)} \\ \xi_{t+p-1,s}^{(1)} & \xi_{t+p-1,s}^{(2)} & \cdots & \xi_{t+p-1,s}^{(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{t+1,s}^{(1)} & \xi_{t+1,s}^{(2)} & \cdots & \xi_{t+1,s}^{(p)} \end{vmatrix}} = (-1)^{p-1} \frac{|\Xi_{t+p-1,s}|}{|\Xi_{t+p,s}|}. \quad (\text{A.8})$$

The Casoratian recurrence in eq. (A.6) takes the form

$$|\Xi_{t+p,s}| = (-1)^{p-1} \phi_p(t+p) |\Xi_{t+p-1,s}| \quad (\text{A.9})$$

Taking into account that  $\phi_p(q) \neq 0$  for all  $q \geq s+1$ , setting  $q = t+p$ , it follows that  $q = t+p \geq s+1$ , whence  $t \geq s+1-p$ . and  $\phi_p(t+p) \neq 0$  for all  $t \in \mathbb{Z}_{s+1-p}$ . We conclude that eq. (A.9) can be equivalently written as:

$$\frac{1}{\phi_p(t+p)} = (-1)^{p-1} \frac{|\Xi_{t+p-1,s}|}{|\Xi_{t+p,s}|}. \quad (\text{A.10})$$

As the right-hand side members of eqs. (A.8) and (A.10) coincide, the result follows.  $\square$

## Appendix B [The Infinite Gaussian Elimination]

We present here the basic steps of the IGE implemented with rightmost pivot elements the  $(-1)$ s in eq. (26). It constructs the rows of  $\mathbf{FRREF}(\mathbf{A})$  along with the particular solution (see Subsection 3.1), yielding equivalent row-recurrences. At the end of this Appendix we give some supplementary results, as reported in Example 1.

In what follows, the rows of the coefficient matrix,  $\mathbf{A}$ , in eq. (26) are denoted as  $\mathbf{R}_i$  for  $i \geq 1$ . In the first algorithmic step  $\mathbf{R}_1$  is normalized by multiplying  $\mathbf{R}_1$  with  $(-1)$ , yielding:  $\tilde{\mathbf{R}}_1 = (-1)\mathbf{R}_1$ . The new row  $\tilde{\mathbf{R}}_1$  replaces  $\mathbf{R}_1$  and remains invariant during the forthcoming process. That is  $\tilde{\mathbf{R}}_1$  is the first row of  $\mathbf{FRREF}(\mathbf{A})$ . In the second step the algorithm uses  $\tilde{\mathbf{R}}_1$  as pivot row to eliminate the entry  $\phi_1(r+2)$  of  $\mathbf{R}_2$ , positioned in the same column and below the (rightmost) pivot entry 1 of  $\tilde{\mathbf{R}}_1$ . This is obtained by multiplying  $(-\tilde{\mathbf{R}}_1)$  (or  $\mathbf{R}_1$ ) with  $\phi_1(r+2)$  and adding the result to  $\mathbf{R}_2$ . After normalization, the second step is described by:  $\tilde{\mathbf{R}}_2 = -[\phi_1(r+2)(-\tilde{\mathbf{R}}_1) + \mathbf{R}_2]$ . The new row  $\tilde{\mathbf{R}}_2$  replaces  $\mathbf{R}_2$ , yielding the second row of  $\mathbf{FRREF}(\mathbf{A})$ . In the third step the algorithm uses  $\tilde{\mathbf{R}}_1$  and  $\tilde{\mathbf{R}}_2$  as pivot rows to eliminate the entries  $\phi_2(r+3)$  and  $\phi_1(r+3)$  of  $\mathbf{R}_3$ , respectively. After normalization, the new row  $\tilde{\mathbf{R}}_3$  is given by:  $\tilde{\mathbf{R}}_3 = -[\phi_2(r+3)(-\tilde{\mathbf{R}}_1) + \phi_1(r+3)(-\tilde{\mathbf{R}}_2) + \mathbf{R}_3]$ . The new row  $\tilde{\mathbf{R}}_3$  replaces  $\mathbf{R}_3$  yielding the third row of  $\mathbf{FRREF}(\mathbf{A})$ . Proceeding in this way the algorithm constructs the rows of a  $\mathbf{FRREF}$  of  $\mathbf{A}$ , which are given by

$$\tilde{\mathbf{R}}_i = \phi_1(r+i)\tilde{\mathbf{R}}_{i-1} + \phi_2(r+i)\tilde{\mathbf{R}}_{i-2} + \cdots + \phi_p(r+i)\tilde{\mathbf{R}}_{i-p} - \mathbf{R}_i \quad (\text{B.1})$$

provided that  $\tilde{\mathbf{R}}_i = \mathbf{0}$ , whenever  $1-p \leq i \leq 0$ . The set  $\{\tilde{\mathbf{R}}_i\}_{1-p \leq i \leq 0}$ , consisting of  $p$  zero rows, can be viewed as a set of initial conditions associated with the recurrence in eq. (B.1), generating the rows of  $\mathbf{FRREF}(\mathbf{A}) = [\tilde{\mathbf{R}}_i]_{i \geq 1}$ . For example if  $i = 1$ , then eq. (B.1) gives:  $\tilde{\mathbf{R}}_1 = -\mathbf{R}_1$ , since  $\tilde{\mathbf{R}}_0 = \tilde{\mathbf{R}}_{-1} = \cdots = \tilde{\mathbf{R}}_{1-p} = \mathbf{0}$ . If  $i = 2, 3$ , then eq. (B.1) gives:

$$\begin{aligned} \tilde{\mathbf{R}}_2 &= \phi_1(r+2)\tilde{\mathbf{R}}_1 - \mathbf{R}_2 = -[\phi_1(r+2)(-\tilde{\mathbf{R}}_1) + \mathbf{R}_2] \\ \tilde{\mathbf{R}}_3 &= \phi_2(r+3)\tilde{\mathbf{R}}_1 + \phi_1(r+3)\tilde{\mathbf{R}}_2 - \mathbf{R}_3 = -[\phi_2(r+3)(-\tilde{\mathbf{R}}_1) + \phi_1(r+3)(-\tilde{\mathbf{R}}_2) + \mathbf{R}_3], \end{aligned}$$

as expected, and so forth.

As an alternative, the recurrence in eq. (B.1) can be viewed as a VC-LDE( $p$ ) with initial condition sequences  $\tilde{\mathbf{R}}_m$  for  $1 \leq m \leq p$ , constructed by the finite Gaussian elimination algorithm after a sequence of  $p$  steps, whereas the algorithm is implemented with rightmost pivoting and forcing terms  $-\mathbf{R}_i$  for  $i \geq 1$ . Thereafter, the recurrence in eq. (B.1) generates the remaining rows:  $\{\tilde{\mathbf{R}}_i\}_{i \geq p+1}$ .

Next, we employ Example 1 to verify the first two zero outcomes of the product  $\mathbf{A}[0, 1, \xi_{r+1,r}^{(1)}, \xi_{r+2,r}^{(1)}, \dots]'$ :

$$\begin{aligned} [\phi_2(r+1), \phi_1(r+1), -1, 0, \dots][0, 1, \xi_{r+1,r}^{(1)}, \xi_{r+2,r}^{(1)}, \dots]' &= \phi_1(r+1) \cdot 1 + (-1)\phi_1(r+1) = 0, \\ [0, \phi_2(r+2), \phi_1(r+2), -1, 0, \dots][0, 1, \xi_{r+1,r}^{(1)}, \xi_{r+2,r}^{(1)}, \dots]' &= \phi_2(r+2) \cdot 1 + \phi_1(r+2)\phi_1(r+1) + \\ &\quad (-1)(\phi_1(r+1)\phi_1(r+2) + \phi_2(r+2)) = 0, \\ &\quad \dots \qquad \qquad \qquad \dots \end{aligned}$$

A particular solution is also constructed by the IGE algorithm, by applying the same sequence of row elementary operations, used by the IGE for the row reduction of  $\mathbf{A}$  to  $\mathbf{FRREF}(\mathbf{A})$ , but now to the sequence  $\{-v_{r+i}\}_{i \geq 1}$ . The algorithm gives rise to a recurrence, which similarly follows as in eq. (B.1)

$$\tilde{v}_{r+i} = \phi_1(r+i)\tilde{v}_{r+i-1} + \phi_2(r+i)\tilde{v}_{r+i-2} + \dots + \phi_p(r+i)\tilde{v}_{r+i-p} + v_{r+i}, \quad i \geq 1,$$

taking on zero initial values, that is  $\tilde{v}_{r+1-m} = 0$  for all  $m \in \llbracket 1, p \rrbracket$ . The so constructed solution sequence  $\{\tilde{v}_t\}_{t \geq r+1}$  is a particular solution, which is also represented explicitly by a Hessenbergian function that is  $y_t^{par} = \tilde{v}_t$  for  $t \geq r+1$  (see Proposition 5).

The general solution of eq. (25) is a linear combination of the fundamental solutions with coefficients arbitrary initial condition values  $y_{r+1-m} = a_m$  for  $1 \leq m \leq p$  (that is the general homogeneous solution, see Proposition 3) plus the particular solution mentioned above (see also eq. (60)).

## Appendix C [Algorithms]

Two Algorithms are presented in this Appendix. The first, returns the Leibnizian representation of Hessenbergians given in eq. (24). The second, returns the restriction of the Green's function  $H(t, r)$ , involved in the general homogeneous solution of VC-LDEs( $p$ ), which coincides with  $\xi_{t,r}$  (see Theorem 5). This algorithm is completed by the construction of the general nonhomogeneous solution of eq. (61), expressed in terms of the Green's function  $H(t, r)$  (or  $\xi_{t,s}$ ). Both Algorithms are followed by automatically executable computer programs written in Mathematica's symbolic language. The instructions of the algorithms follow the structure of the paper and use the corresponding formulas established in it. This can be viewed as a verification scheme for the validity of the results derived and used in the paper.

In order to run the first program one needs to insert the order  $k$  of the matrix. This is the only one external input, whereas the other inputs are internal instructions defined within the program and remain invariant in each new call of the program. Using Mathematica symbolic computation, the program returns an expression of eq. (24) exclusively in terms of the non-trivial entries  $h_{i,j}$  of  $\mathbf{H}_k$ . This program is to be compared with corresponding routines evaluating Hessenbergians.

The functions  $\mathfrak{z}_{k,i}(\mathbf{r})$  and  $\tau_k(m)$  along with their composite  $\sigma_{k,i}(m)$  are defined within the program expressing the corresponding formulas in the chosen language. In their program notation, the variable  $k$  (the order of the matrix) is omitted. Instead they are designated as  $\mathfrak{z}_i, \tau, \sigma_i$ , respectively, since all these functions are redefined for each new input of  $k$ .

In both algorithms each algorithmic step is followed by the corresponding instruction of the program, which is directly executable by Mathematica.

**Algorithm 1** (Leibnizian representation of Hessenbergians).

*In*[1] : \$Assumptions =  $k > 0$  &&  $k \in \text{Integers}$ ;

i) Enter the order of the Hessenberg matrix:

*In*[2] :  $k := \dots$

ii) Define the Hessenberg matrix  $\mathbf{H}_k = (h_{i,j})_{i,j \in \llbracket 1, k \rrbracket}$  of order  $k$ :

*In*[3] :  $\mathbf{H}[k] := \text{Table}[\text{If}[j \leq i + 1, h[i, j], 0], \{i, 1, k\}, \{j, 1, k\}]$

iii) Define the entries  $c_{i,j}$  of  $\mathbf{H}_k$ , according to eq. (5):

*In*[4] :  $c[i_, j_] := \text{If}[j \neq i + 1, \mathbf{H}[k][[i, j]], -\mathbf{H}[k][[i, j]]]$

iv) Define the  $i$ th component, say  $\tau_i(m)$ , of  $\tau(m)$  (given by eq. (19)) and assign  $\tau_1(m) = 1$ , whenever  $i \notin \llbracket 1, k - 1 \rrbracket$ :

*In*[5] :  $\tau[i_, m_] := \text{If}[1 \leq i \leq k - 1, \lfloor m \div 2^{k-i-1} \rfloor - 2 \lfloor \frac{\lfloor m \div 2^{k-i-1} \rfloor}{2} \rfloor, 1]$

- v) Define the composition of  $\mathfrak{z}_i$  in eq. (14) and  $\tau$  in eq. (19), using the function  $\zeta_i$  in eq. (12), which, in turn, is constructed in a step by step procedure as follows:

a) Define the list of products  $(j \cdot \tau(j, m))_{j=0,1,\dots,i-1}$ :

$$In[6] : \text{Prod}[i\_ , m\_ ] := \text{Table}[ j \times \tau[j, m], \{j, 0, i - 1\}]$$

b) Evaluate the maximum value of  $\text{Prod}[i, m]$  and group these values in lists  $M[m] = \{\max(\text{Prod}[i, m], i \in \llbracket 1, k \rrbracket)\}$ :

$$In[7] : M[m\_ ] := \text{Table}[ \text{Max}[\text{Prod}[i, m]], \{i, 1, k\}]$$

c) Define the function  $Z(i, m) = \zeta_i \circ \tau(m)$  for  $(i, m) \in \llbracket 1, k \rrbracket \times \mathbb{I}_{k-1}$ , according to eq. (12), that is  $Z(i, m)$  is the number of consecutive zero predecessors of the factor  $\tau(i, m)$  :

$$In[8] : Z[i\_ , m\_ ] := \tau[i, m] \times (i - M[m][[i]]) - 1$$

vi) Define  $\sigma_i(m)$  in terms of  $Z(i, m)$ , defined as  $\sigma_i(m) = \mathfrak{z}_i \circ \tau(m) = i - \zeta_i(\tau(m))$ :

$$In[9] : \sigma[i\_ , m\_ ] := i - Z[i, m]$$

vii) Define the Hessenbergian formula (24):

$$In[10] : \text{Hsb}[k] := \sum_{m=0}^{2^{k-1}-1} \prod_{i=1}^k c[i, \sigma[i, m]]$$

viii) Expand the Hessenbergian formula:

$$In[11] : \text{Expand}[\text{Hsb}[k]]$$

ix) Check whether the equation  $\text{Hsb}[k] = \text{Det}[\mathbf{H}[k]]$  holds true, where  $\text{Det}[\ ]$  stands for Mathematica's symbolic evaluation of determinants:

$$In[12] : \text{Hsb}[k] - \text{Det}[\mathbf{H}[k]] == 0$$

As an example, setting  $k = 4$  and running the above program, it returns  $\text{Hsb}[4]$ :

$$\begin{aligned} Out[1] = & -h[1, 2]h[2, 3]h[3, 4]h[4, 1] + h[1, 1]h[2, 3]h[3, 4]h[4, 2] \\ & + h[1, 2]h[2, 1]h[3, 4]h[4, 3] - h[1, 1]h[2, 2]h[3, 4]h[4, 3] \\ & + h[1, 2]h[2, 3]h[3, 1]h[4, 4] - h[1, 1]h[2, 3]h[3, 2]h[4, 4] \\ & - h[1, 2]h[2, 1]h[3, 3]h[4, 4] + h[1, 1]h[2, 2]h[3, 3]h[4, 4] \end{aligned}$$

The program replies to the instruction (ix): "TRUE". This can be repeated by any value of  $k \geq 2$ , and the program replies "TRUE".

The second Algorithm computes the Green's function restriction stated in Theorem 5 as a Hessenbergian, followed by the construction of the general solution of eq. (25) in terms of the Green function in eq. (61).

**Algorithm 2** (Green's function and the general solution of VC-LDEs( $p$ )).

$$In[1] : \$Assumptions = p > 0 \ \&\& \ p \in \text{Integers} \ \&\& \ s \in \text{Integers} \ \&\& \ r \in \text{Integers} \ \&\& \ t \in \text{Integers};$$

i) Enter the order of the linear difference equation:

$$In[2] : p := \dots$$

ii) Enter the value of the variable  $r \geq s$ :

$$In[3] : r := \dots$$



iii) Enter the value of the variable  $t$  such that  $t > r$ :

$$\text{In}[4] : t := \dots$$

iv) Replace the entries of  $\mathbf{H}_k$  with the entries of  $\Phi_{t,r}$  according to the assignment 64:

$$\text{In}[5] : h[i\_ , j\_ ] := \text{Which}[i == j - 1, -1, 1 \leq i - j + 1 \leq p, \phi_{i-j+1}[r + i], \text{True}, 0]$$

v) Define the principal matrix as a function of  $n, m$  for  $n > m$ :

$$\text{In}[6] : \Phi[n\_ , m\_ ] := \text{Table}[h[i, j], \{i, m + 1 - r, n - r\}, \{j, m + 1 - r, n - r\}]$$

vi) Define the Green's function restriction:

$$H(n, j) = \xi_{n,j} \text{ for } n > j, H(n, j) = 0 \text{ for } n < j, H(n, j) = 1 \text{ for } n = j:$$

$$\text{In}[7] : H[n\_ , j\_ ] := \text{Which}[n < j, 0, n == j, 1, n == j + 1, \phi_1[j + 1], n \geq j + 2, \text{Det}[\Phi_{n,j}]]$$

vii) Define the general solution formula in eq. (61) (or eq. (60)) with initial condition values  $\{y_{r-p+1}, \dots, y_r\}$ , and forcing terms  $v_{r+i}$ , as a function of  $n$ :

$$\text{In}[8] : y[n\_ ] := \sum_{m=1}^p \sum_{i=1}^{p+1-m} \phi_{m-1+i}(r+i)H(n, r+i)y_{r+1-m} + \sum_{i=1}^{n-r} H(n, r+i)v_{r+i}.$$

viii) Apply the Definition of the Green's function in (vi) for  $n = t$  and  $j = r$ :

$$\text{In}[9] : \text{Expand}[H[t, r]]$$

ix) Apply the general solution with  $n = t$ :

$$\text{In}[10] : \text{Expand}[y[t]]$$

As an illustrative example, setting  $p = 2, t = 5, s = 1$  and  $r = 2$  and running the above program, it returns the following expression

$$\text{Out}[1] := \phi_1(3)\phi_1(4)\phi_1(5) + \phi_1(5)\phi_2(4) + \phi_1(3)\phi_2(5).$$

This is an expansion of the Green's function  $H(5, 2) = \xi_{5,2}$  associated with the second order VC-LDE.

The solution  $y_5$  of the initial value problem  $y_2 = a, y_1 = b$  with forcing terms  $v_3, v_4, v_5$  is also recovered by the program yielding:

$$\begin{aligned} \text{Out}[2] := & \phi_1(3)\phi_1(4)\phi_1(5)a + \phi_2(4)\phi_1(5)a + \phi_1(3)\phi_2(5)a + \phi_1(4)\phi_1(5)\phi_2(3)b + \phi_2(3)\phi_2(5)b \\ & + v_4\phi_1(5) + v_3\phi_1(4)\phi_1(5) + v_3\phi_2(5) + v_5. \end{aligned}$$

This result is in accord with the solution expansion obtained directly by recursion.