Conventional Inference in the Vicinity of Refined Nonstationarity Boundaries: Regressions with Heavy Tailed Weakly Nonstationary Processes

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Introduction

- Estimation and inference in parametric & non-parametric regressions of the form:
  - $y_t = \mu + \beta f\left(x_{t-1}\right) + u_t,\ f(.)$ known (parametric)
  - $y_t = f\left(x_{t-1}\right) + u_t,\ f(.)$ unknown (non-parametric estimation)
- New limit theory for x_t **HT-WNPs** i.e. *heavy tailed weakly nonstationary process*.
- Conventional inference for **HT-WNPs**

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- Estimation and inference in parametric & non-parametric regressions of the form:
  - $y_t = \mu + \beta f\left(x_{t-1}\right) + u_t$, $f(.)$ known (parametric)
  - $y_t = f\left(x_{t-1}\right) + u_t$, $f(.)$ unknown (non-parametric estimation)
- New limit theory for x_t **HT-WNPs** i.e. *heavy tailed weakly nonstationary process*.
- Conventional inference for **HT-WNPs** but also for
  - **HT-SP** (HT Stationary Processes)
  - **HT-SNP** (HT Stronlgy Nonstationary Processes)
- u_t is a m.d. error **finite second moment (main assumption)**
  - some limit results for HT regression error also provided

Introduction

- x_t is driven by innovations v_t (linear process) in the *domain of attraction of an α-stable law* $(0 < \alpha \leq 2)$.
- $\alpha \in (0, 2]$ tail parameter
- For $\alpha < 2$: Any $\alpha' < \alpha$ moment exists - **finite sub-α moments**
- For $\alpha = 2$ the second moment may or may not exist.
- If a second moment exists, $\alpha = 2$.
- More details later

Introduction

Examples of HT time series x_t on innovations v_t (v_t HT linear process):

- **HT-WNPs** (weakly nonstationary processes):
  - Fractional $I(d)$, $d = 1 - \frac{1}{\alpha}$
  - Mildly Integrated (Phillips & Magdalinos, 2007)

$$x_t = \rho_n x_{t-1} + v_t,\ \rho_n = 1 - \kappa_n,\ \kappa_n / n \to 0$$

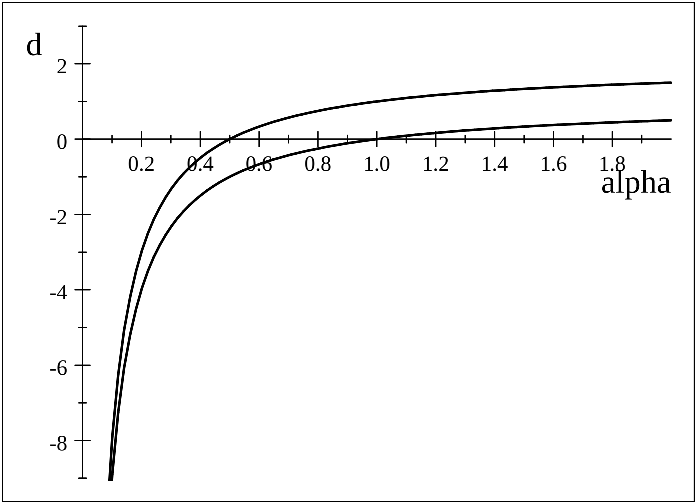
- **HT-SNPs** (strongly nonstationary processes):
  - $I(d)$, $1 - \frac{1}{\alpha} < d < \frac{3}{2} - \frac{1}{\alpha}$
  - NI, $x_t = (1 + \frac{c}{n})x_{t-1} + v_t$ (with v_t possibly fractional)

- **HT-SPs** (stationary processes)
  - $I(d)$, $d < 1 - \frac{1}{\alpha}$

Introduction

- If v_t has a second moment $\Rightarrow \alpha = 2$
- **WNPs:** $I(d)$, $d = 1/2$
- **SNPs:** $I(d)$, $1/2 < d < 3/2$
- **SPs**: $I(d)$, $d < 1/2$

Introduction



Introduction

- Data Consistent with HT behaviour: Review in Embrechts, Klüppelberg & Mikosch (1997), Alder, Feldman & Taqqu (1998), Ibragimov, Ibragimov & Walden (2015)
  - actuarial science
  - economics
  - engineering (telecommunications network traffic)
  - finance
  - meteorology
  - urban studies
  - epidemiology; e.g. Cohen, Davis & Samorodnitsky (2022): COVID-19 deaths in US exhibit infinite variance.

Introduction: Examples in Economics

- **Exchange rates** and **exchange rate returns** in emerging markets: e.g. Falk and Wang (2003), Ibragimov, Ibragimov & Walden (2015), Nicolau & Rodrigues (2019), Barigozzi, Cavaliere & Trapani (2021)
- **Inflation**: Charemza, Hristova & Burridge (2006): **monthly CPI/RPI Tail estimates ($\widehat{\alpha}$)**

Country		Price Innovations	Inflation Innovations
Mexico	1950-2000	1.27	1.28
Spain	1957-2000	1.43	1.40
UK	1956-2000	1.48	1.58
Canada	1950-2000	1.77	1.78
France	1957-2000	1.76	1.76
USA	1950-2000	1.90	1.91
Belgium	1957-2000	1.90	1.83

Introduction: Examples in Economics

- **Commodity Prices**: Barigozzi, Cavaliere & Trapani (2021): Copper, Gold Brent crude, Dubai Crude, Nickel, Cobalt

$$H_0 : \ Var(X) = \infty \text{ not rejected}$$

- **Realised Stock Return Volatility**: Kim & Meddahi (2020); Kim, Meddahi & Yamashita (2021): Evidence for infinite variance, in particular during crisis periods.

Introduction

- Challenges/Objectives:
  1) Characterise limit distributions when data are possibly nonstationary and HT:
     - FCLTs in general fail in this framework
     - This is the case for WNPs irrespective of the HT characteristics of data
     - Interestingly, if $\alpha < 2$, FCLTs fail for SNPs for all $d < 1$.

[more results on SNPs: Kasparis, Phillips & Wang (2023)]

- 2) Obtain conventional inference, free of nuisance parameters:
  - d (memory)
  - α (tail)
  - c (near-to-unity)

Introduction

- long memory + HTs $\Rightarrow$ enlargement of the nonstationary region
  - i.e. the model space over which conventional inference is not in general applicable
  - Stationarity boundary (fractional case)

$$d = 1 - \frac{1}{\alpha}, \; \alpha \in (0, 2]$$

  - For $\alpha = 2 \Rightarrow d = 1/2$
  - For $\alpha < 2 \Rightarrow d < 1/2$
  - For $\alpha < 1 \Rightarrow d < 0$
- In general, inferential methods that are applicable in a stationary framework are not applicable under nonstationarity and vice versa.
- Early methods that (e.g. FMLS) for nonstationary models are not robust to local deviations from unit roots.

Introduction: Robust Methods

- **Conservative inferential methods**: Cavanaght, Elliot & Stock (1995), Campbell & Yogo (2006), Mikusheva (2007), Elliott, Müller & Watson (2015)
  - e.g. Bonferroni methods or test statistics averaged over a prespecified range for the nuisance parameter space
  - In general valid under near-unit roots and in some cases under larger deviations
  - More complicated test statistics/non standard limit distributions

Introduction: Robust Methods

- **Estimators with mixed normal distributions:** conventional inference/ self-normalised statistics
- **Fractional Systems**: Robinson & Hualde (2003), Christensen & Nielsen (2006), Hualde & Robinson (2010), Andersen & Varneskov (2020).
  - Systems of equations
  - do not allow for near unit roots/ require memory estimators
  - (mostly) parametric convergence rates
- **Semi-parametric IV**
  - IVX: Magdalinos & Phillips (2009), Kostakis, Magdalinos & Stamotogiannis (2015);
  - CTLS: Hu, Kasparis & Wang (2022);
  - Smooth IV: Kasparis, Phillips & Wang (2023)
- **non-parametric**: Wang & Phillips (2009a,b), Kasparis et al. (2015), Duffy & Kasparis (2021, 2023)

Introduction: Autoregressions HT data (stationary)

- Nuisance (tail) parameters when regression is HT
  - e.g. Davis & Resnick (1985, 1986), Davis, Knight & Liu (1992)
- Weighted LAD: e.g. Pan, Wang & Yao (2007), Zhu & Ling (2012).
  - Gaussian limit distributions
- Bootstap methods: e.g. Cavaliere, Georgiev and Taylor (2016)

Introduction: Autoregressions HT data (nonstationary)

- Most results concern unit root models i.e. $d = 1$
- Recall that FCLTs do not apply for HT-SNPs for $d < 1$, $\alpha < 2$
- Unit Root Autoregressions
  - e.g. Chan & Tran (1989), Knight (1989), Phillips (1991), Caner (1997, 1998)
- Non standard distrbutions that involve nuisance (tail) parameters
  - Cavaliere, Georgiev & Taylor (2018): bootstrap methods
  - Hallin, van den Akker & Werker (2011): distribution free test statistics
- Barigozzi, Cavaliere & Trapani (2021): Randomised test for cointegrating rank

Stable distributions

- A r.v. ξ has an $a-$stable distribution ($S_\alpha(\sigma, \beta, \mu)$), if

$$\mathbf{E}e^{i\lambda\xi} = \begin{cases} \exp\left\{-\sigma\left|\lambda\right|^{\alpha}\left[1 - i\beta\left(\text{sign}\lambda\right)\tan\frac{\pi\alpha}{2}\right] + i\mu\lambda\right\}, & \alpha \neq 1 \\ \exp\left\{-\sigma\left|\lambda\right|^{\alpha}\left[1 - i\beta\frac{2}{\pi}\left(\text{sign}\lambda\right)\ln\left|\lambda\right|\right] + i\mu\lambda\right\}, & \alpha = 1 \end{cases}$$

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- A r.v. ξ in *the domain of attraction of an $a-$stable distribution* ($\tilde{S}_\alpha(\sigma,\beta,\mu)$), if as $\lambda \to 0$

$$\mathbf{E}e^{i\lambda\xi} = \exp\left\{\left[-\sigma\left|\lambda\right|^{\alpha} H\left(\left|\lambda\right|^{-1}\right)\left[1 - i\beta\left(\mathrm{sign}\lambda\right)\tan\frac{\pi\alpha}{2}\right] + i\mu\lambda\right] \times \left[1 + o(1)\right]\right\}, \ \alpha \neq 1$$

  - H is SV
  - a similar expression holds for $\alpha = 1$

- $\xi \sim \tilde{S}_\alpha(\sigma, \beta, \mu)$, $\alpha \in (0, 2)$ iff $(x \to \infty)$

$$\mathbf{P}(\xi > x) = (c_1 + o(1))\, x^{-\alpha} h(x), c_1 \geq 0,\ \ h \sim SV$$

  and

$$\mathbf{P}(\xi < -x) = (c_2 + o(1))\, x^{-\alpha} h(x), c_2 \geq 0,\ \ h \sim SV$$

- Set (strictly stable)

$$\begin{aligned} S_\alpha(\sigma, \beta) \quad &: \quad = S_\alpha(\sigma, \beta, 0) \\ \tilde{S}_\alpha(\sigma, \beta) \quad &: \quad = \tilde{S}_\alpha(\sigma, \beta, 0) \end{aligned}$$

Examples of WNPs:

- Fractional $d = 1 - 1/\alpha$

e.g. *ARFIMA Type-I*

$$x_t = x_{t-1} + v_t, \;\; x_0 = O_p(1)$$

- $v_t \sim ARFIMA(\delta)$ Type-I
- $d = 1 + \delta$

e.g. *ARFIMA Type-II:*

$$(I - L)^d x_t = v_t 1\{t > 0\}$$

- $v_t \sim$ short memory linear process e.g. ARMA

Examples of WNPs:

- Mildly Integrated processes (e.g. Phillips & Magdalinos, 2007): $\{x_t\}_{t=1}^{n}$

$$x_t = \rho_n x_{t-1} + v_t, \ \ x_0 = O_p(1)$$

  - $\rho_n = 1 - \kappa_n^{-1}$, $\kappa_n = o(n)$
  - v_t stationary HT-LP short memory or long memory (not considered here)

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Limit Theory for HT-WNPs

- Duffy & Kasparis (2021): WNPs under **second moments**
- WNPs processes do not satisfy FCLTs (even if a second moment exists)
- $x_t\left(n\right) = \sum_{k=0}^{\infty} a_{k,t}\left(n\right)\xi_{t-k} =: x_t^{+}\left\{\xi_t, ..., \xi_1\right\} + x_t^{-}\left\{\xi_0, \xi_{-1}, ...\right\}$
- Set $X_n(s) := \gamma_n^{-1} x_{\lfloor ns \rfloor}$, γ_n a suitable normalising sequence ($\gamma_n \to \infty$)
  - e.g. $\gamma_n = Var(x_n)^{1/2}$, if second moments exist

Limit Theory for HT-WNPs

- $X_n^+(s) := \gamma_n^{-1} x_{\lfloor ns \rfloor}^+$ and $X_n^-(s) = \gamma_n^{-1} x_{\lfloor ns \rfloor}^-$
- It can be shown that for $s \in (0, 1]$

$$X_n^+(s) \xrightarrow{fdd} X^+(s)$$

  - such that $X^+(r) \perp X^+(s)$, $X^+(r) =_d X^+(s)$ (*iid process*)
- Further

$$\left[X_n^-(r), X_n^-(s)\right] \xrightarrow{d} \left[X^-, X^-\right]$$

Limit Theory for HT-WNPs: Locally Integrable Functions

- Suppose that
  - $|f(x)| \leq C\left(1+|x|^{\alpha'}\right)$, $\alpha' \in (0, \alpha)$
  - x_t-$\mathcal{F}_t$ measurable, HT-WNP ($S_\alpha(\sigma, \beta)$ innovations);
  - $\{u_t, \mathcal{F}_t\}$ m.d., $\mathbf{E}\left(u_t^2 \mid \mathcal{F}_{t-1}\right) = \sigma_u^2 < \infty$.

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- Then under some additional technical conditions,

$$\frac{1}{n}\sum_{t=1}^{n} f\left(\gamma_n^{-1}x_t\right) \xrightarrow{d} \int_{\mathbb{R}} f\left(x+X^-\right)\phi_{X^+}(x)dx,$$

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} f\left(\gamma_n^{-1}x_{t-1}\right)u_t \xrightarrow{d} MN\left(0, \sigma_u^2\int_{\mathbb{R}} f\left(x+X^-\right)^2\phi_{X^+}(x)dx\right),$$

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where ϕ_{X^+} is the density of X^+ and

$$\left[X^{+},X^{-}\right] \sim \begin{cases} \left[\ S_\alpha(\sigma\alpha^{\alpha+1},\beta), \quad S_\alpha(\sigma\alpha^{\alpha+1},\beta)\ \right] & \text{FR type I} \\ \left[\ S_\alpha\left(\sigma\alpha\left|\sum_{j=0}^{\infty}c_j\right|^{\alpha},\beta\right), \quad 0\ \right] & \text{FR type II} \\ \left[\ S_\alpha\left(\sigma\left|\sum_{j=0}^{\infty}c_j\right|^{\alpha},\beta\right), \quad 0\ \right] & \text{MI} \end{cases}$$

Limit Theory for HT-WNPs

- **Remark.**
  - **Fractional**: γ_n is the *unique SV sequence* that satisfies
  $$\gamma_n^{\alpha} \sim \int_1^{n^{1/\alpha}\gamma_n} \frac{H(u)}{u} du$$
  - **MI**: $\gamma_n = \kappa_n^{1/\alpha} \ell(\kappa_n)$, where ℓ is the *unique SV function* that satisfies
  $$\ell(x)^{\alpha} \sim H(x^{1/\alpha}\ell(x))$$

Limit Theory for HT-WNPs

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$$\ell(x)^{\alpha} \sim H(x^{1/\alpha}\ell(x))$$

- **Remark**. *Asymptotically Homogeneous Function* (**AHF**) f of order $k_H(.)$ (e.g. Park & Phillips, 2001)
$$f(\lambda x) \approx k_H(\lambda) f_H(x), \text{ as } \lambda \to \infty,$$
Then
$$\frac{1}{nk_H(\gamma_n)} \sum_{t=1}^{n} f(x_t) \xrightarrow{d} \int f_H(x + X^-)\phi_{X^+}(x)dx$$
- e.g. polynomial functions, logarithmic, distribution type, indicator.

Limit Theory for HT-WNPs: Kernel Functions

- Suppose that:
  - $\int |K| + K^2 < \infty$ and $\frac{\gamma_n}{h_n n} + \frac{h_n}{\gamma_n} \to 0$;
  - x_t-$\mathcal{F}_t$ measurable, HT-WNP;
  - $\{u_t, \mathcal{F}_t\}$ m.d., $\mathbf{E}\left(u_t^2 \mid \mathcal{F}_{t-1}\right) = \sigma_u^2 < \infty$.

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- Then under some additional technical conditions,

$$\frac{\gamma_n}{h_n n} \sum_{t=1}^{n} K\left(\frac{x_t - x}{h_n}\right) \xrightarrow{d} \phi_{X^+}\left(-X^-\right) \int_{\mathbb{R}} K(u)du,$$

$$\sqrt{\frac{\gamma_n}{h_n n}} \sum_{t=1}^{n} K\left(\frac{x_{t-1} - x}{h_n}\right) u_t \xrightarrow{d} MN\left(0, \sigma_u^2 \phi_{X^+}\left(-X^-\right) \int_{\mathbb{R}} K(u)^2 du\right).$$

Stationary Case Comparison

- Let $x_t \sim I(d)$, $d \in (-\infty, 1-\frac{1}{\alpha})$

$$x_t = \sum_{j=0}^{\infty} c_j \xi_{t-j}, \, c_j \sim j^{d-1}, \, \xi_t \sim id.S_\alpha \left\{ \sigma^\alpha, \beta \right\}.$$

  - If $\sum_{j=0}^{\infty} \left| c_j \right|^{\tau \wedge 1} < \infty$, $\tau \in (0, \alpha)$, (Davis & Resnick, 1985)

$$\frac{1}{a_n^2} \sum_{t=1}^{n} x_t^2 \xrightarrow{d} \sum_{j=0}^{\infty} c_j^2 \cdot X_0, \quad \text{with } a_n^{-2} \sum_{t=1}^{n} \xi_t^2 \xrightarrow{d} X_0$$

$(a_n \to \infty)$

  - If $\mathbf{E} \left| f(x_t) \right| < \infty$, by ergodicity

$$\frac{1}{n} \sum_{t=1}^{n} f(x_t) \xrightarrow{L_1} \mathbf{E} f(x_t).$$

  - If ξ_t has a bounded density (Wu & Mielniczuk, 2002)

$$\frac{1}{h_n n} \sum_{t=1}^{n} K\left(\frac{x_t - x}{h_n} \right) \xrightarrow{p} \phi_{x_1}(0) \int_{\mathbb{R}} K(x) dx, \; \phi_{x_1}(.) \text{ density of } x_1.$$

Nonstationary Case Comparison

- Let $x_t \sim I(d)$, $1 - 1/\alpha < d < 3/2 - 1/\alpha$
  - Astruaskas (1983), Kasahara & Maejima (1988)

$$\gamma_n^{-1} x_{[nr]} \overset{f.d.d.}{\to} \Lambda_{\alpha,d}(r) = \begin{cases} \int_{-\infty}^{t} \left[(t-r)^{d-1} - (-r)^{d-1} 1_{\{r<0\}} \right] dZ_\alpha(r), \\ Z_\alpha(r), \ d = 1 \end{cases}$$

  $Z_\alpha(r)$ Lévy process ($\gamma_n = n^H \ell_n, H = d - 1 + 1/\alpha$, ℓ_n SV).

  - $\Lambda_{\alpha,d}(r) \notin D[0,1]$ for $1 - 1/\alpha < d < 1$, $\alpha < 2$
  - Astrauskas (1983), Avram & Taqqu (1992): convergence in $D[0,1]$ for $d \geq 1$
  - Jeganathan (2004):

$$\frac{\gamma_n}{h_n n} \sum_{t=1}^{n} K\left(\frac{x_t - x}{h_n}\right) \overset{d}{\to} L_{\Lambda_{\alpha,d}}(0,1) \int_{\mathbb{R}} K(u) du, \ \Lambda_{\alpha,d} \text{ local time}$$

  - Further results for HT-SNPs: Kasparis, Phillips & Wang (2023)

Parametric Estimation

- Model $y_t = \mu + \beta f(x_{t-1}) + u_t$
- f locally integrable (known) & AHF:

$$f(\lambda x) \simeq \kappa(\lambda) f_H(x),\ \lambda \to \infty$$

- Consider the WLS/IV estimator

$$\tilde{\beta}_{WLS} = \left[\sum_{t=1}^{n} \mathbf{w}_{t-1} \mathbf{x}'_{t-1} \right]^{-1} \left[\sum_{t=1}^{n} \mathbf{w}_{t-1} y_t \right],$$

$\mathbf{x}'_t = [1, f(x_t)]$ and $\mathbf{w}'_t = [w_0(x_t), w_1(x_t)]$ -see e.g. Samorodnitsky, Kurz-Kim & Rachev (2007).

- Weights $|f(x) w_i(x)| = O\left(|x|^{\lambda}\right),\ \lambda \in (0, \alpha)$
  - functions of reduced growth rate facilitate our limit theory
  - yield mixed normal limit distributions i.e. conventional inference

Parametric Estimation: Locally Integrable Weights

- Under certain technical conditions

$$\sqrt{n} diag\{1, \kappa_f(\gamma_n)\}\left(\tilde{\boldsymbol{\beta}}_{WLS} - \boldsymbol{\beta}\right) \xrightarrow{d} \left[\int_{\mathbb{R}} \mathbf{A}(x + X^-)\phi_{X^+}(x)dx\right]^{-1}$$

$$\cdot MN\left[0, \sigma_u^2 \int_{\mathbb{R}} \mathbf{W}(x)(x + X^-)\phi_{X^+}(x)dx\right] \quad (1)$$

- $\mathbf{A}(x) := \begin{bmatrix} w_0(x) & (w_0 \cdot f)(x) \\ w_1(x) & (w_1 \cdot f)(x) \end{bmatrix}$
- $\mathbf{W}(x) := \begin{bmatrix} w_0^2(x) & (w_0 \cdot w_1)(x) \\ (w_0 \cdot w_1)(x) & w_1^2(x) \end{bmatrix}$

Example 1 (Cauchy estimator). Let $f(x) = x$, $w_1(x) = \text{sgn}(x)$, $w_0(x) = 1$.

- Then for all $1 < \alpha \leq 2$.

$$\sqrt{n}\gamma_n \left(\tilde{\beta}_{1,WLS} - \beta_1\right) \xrightarrow{d}$$

$$MN\left[0, \frac{\sigma_u^2 \left\{1 - \left[\int_{\mathbb{R}} \text{sgn}(x + X^-)\phi_{X^+}(x)dx\right]^2\right\}}{\left[\int_{\mathbb{R}} |(x + X^-)| \phi_{X^+}(x)dx - X^- \int_{\mathbb{R}} \text{sgn}(x + X^-)\phi_{X^+}(x)dx\right]^2}\right]$$

Example 2. Let $f(x) = x, w_1(x) = \text{sgn}(x)|x|^{-b_1}$, $w_0(x) = \text{sgn}(x)|x|^{-b_0}$ with $0 < 2b_1, 2b_0 < 1$, $b_1 \neq b_0$.

- Then (1) holds for all $1 - \min(b_1, b_0) < \alpha \leq 2$.

Parametric Estimation: Integrable Weights

- Under certain technical conditions

$$\sqrt{n/\gamma_n}\left(\tilde{\boldsymbol{\beta}}_{WLS} - \boldsymbol{\beta}\right)$$

$$\xrightarrow{d} \left[\int_{\mathbb{R}} \mathbf{A}(x)dx\right]^{-1} \cdot MN\left[0, \sigma_u^2 \phi_{X^+}^{-1}(X^-) \int_{\mathbb{R}} \mathbf{W}(x)dx\right]$$

- t-statitistics based on WLS have $N(0,1)$ limit distributions
- In particular, with **integrable weights**, standard limit theory applies even if x_t ∼**HT-SNP** (cf. Jeganathan, 2004)

Kernel Regression

- Under certain technical conditions

$$\sqrt{\frac{h_n n}{\gamma_n}}\left(\hat{f}(x) - f(x)\right) \xrightarrow{d} MN\left(0, \frac{\sigma_u^2 \int K(s)^2 ds}{\phi_{X^+}(-X^-)\left[\int K(s) ds\right]^2}\right)$$

- Again limit distribution is **MN** even the **SNP** case (provided $\frac{h_n n}{\gamma_n} \to \infty$)

Robust Parametric Specification Test

- Consider the hypothesis (see also Duffy & Kasparis, 2021)

$$H_0 : m(x) = \beta_0 + \beta_1 f(x), \text{ for some } \beta_0, \beta_1 \in \mathbb{R} \text{ and all } x \in \mathcal{X}, \quad (2)$$

the kernel estimator (zero "bias" under H_0)

$$\hat{m}_f(x) = \arg\min_a \min_b \sum_{t=1}^{n} \{y_t - a - b[f(x_{t-1}) - f(x)]\}^2 K_{n,t-1}(x)$$

Robust Parametric Specification Test

- t-statistic: $\tilde{m}_{WLS}(x) := \tilde{\beta}_{0,WLS} + \tilde{\beta}_{1,WLS} f(x)$

$$\hat{t}\left[x, \tilde{m}_{WLS}(x)\right] = \frac{\hat{m}_f(x) - \tilde{m}_{WLS}(x)}{s.e.\left[\hat{m}_f(x) - \tilde{m}_{WLS}(x)\right]},$$

$$\hat{t}\left[x, \tilde{m}_{WLS}(x)\right] = \frac{\hat{m}_f(x) - m(x)}{s.e.\left[\hat{m}_f(x)\right]} + \overbrace{R_n}^{=o_p(1)} \xrightarrow{d} N(0,1)$$

- Duffy & Kasparis 2021: $\frac{\hat{m}_f(x) - \tilde{m}_{OLS}(x)}{s.e.\left[\hat{m}_f(x)\right]}$

Robust Parametric Specification Test

- F-statistic

$$\hat{F} = \sum_{x \in \mathcal{X}} \hat{t}\left[x, \tilde{m}_{WLS}(x)\right]^2, \; \mathcal{X} \text{ discrete set.}$$

- If $\tilde{m}_{WLS}$ is based on integrable weights, we have conventional inference even if x_t is HT-SNP

- Under general conditions, for HT-SP, WNP & SNP:

$$\hat{F} \xrightarrow{d} \chi^2_q, \; q = \#\mathcal{X}$$

Returns-Risk Relationship

- Welch & Goyal, 2018 dataset
- Estimates
  - Memory (d) - Kokoszka & Taqqu (1996)
  - Tail (α) - Hill (2010)
  - Stationarity Threshold $(1 - 1/\alpha)$
- Returns: stationary, finite variance

Monthly Returns		
$\hat{d}_{LW} = 0.07$	$\hat{d}_{ELW} = 0.069$	
$\hat{\alpha}_1 = 2.96$	$\hat{\alpha}_2 = 2.83$	$\hat{\alpha}_3 = 2.58$

Returns volatility: **Highly HT-SNP!**

SVAR$_t$ (Level)		
$\hat{d}_{LW} = 0.53$	$\hat{d}_{ELW} = 0.53$	
$\hat{\alpha}_1 = 1.35$	$\hat{\alpha}_2 = 1.42$	$\hat{\alpha}_3 = 1.24$
$1 - \frac{1}{\hat{\alpha}_1} = 0.26$	$1 - \frac{1}{\hat{\alpha}_2} = 0.29$	$1 - \frac{1}{\hat{\alpha}_3} = 0.19$

ΔSVAR$_t$		
$\hat{d}_{LW} = -0.43$	$\hat{d}_{ELW} = -0.45$	
$\hat{\alpha}_1 = 1.46$	$\hat{\alpha}_2 = 1.24$	$\hat{\alpha}_3 = 1.14$
$1 - \frac{1}{\hat{\alpha}_1} = 0.32$	$1 - \frac{1}{\hat{\alpha}_2} = 0.19$	$1 - \frac{1}{\hat{\alpha}_3} = 0.12$

Rebalancing Memory & Tails in Predictive Regression

- A linear relationship between returns and risk measures is implausible
  - misbalanced both in terms of memory and tails
- Previous work:
  - balance memory using fractionally differenced predictors -Christensen & Nielsen (2007), Bollerslev et al. (2013), Andersen & Varneskov (2021):

$$y_t = \hat{\mu} + \hat{\beta}\Delta^{\hat{d}} x_{t-1} + \hat{u}_t$$

  - balance memory using nonlinear specifications -Marmer (2007), Kasparis (2011), Kasparis et al. (2015), Phillips (2015)

$$y_t = \hat{\mu} + \hat{\beta} f(x_{t-1}) + \hat{u}_t$$

  - Linear filtering cannot rebalance tails

- **To be done**:
- Test the predictability hypothesis using $\hat{F}$

$$H_0 : m(x) = \beta_0 \text{ for some } \beta_0 \in \mathbb{R} \text{ and all } x \in \mathcal{X}$$

- Alternatively use fully nonparametric estimates for testing

$$H_0 : m(x_i) = m(x_j) \text{ for all } x_i, x_j \in \mathcal{X}$$

Thank you!